Algorithms CMSC-37000 First Quiz. January 22, 2009
Instructor: László Babai

Name: ______________________________

Show all your work. **Do not use book, notes, or scrap paper.** Write your answers in the space provided. You may **continue on the reverse.** When describing an algorithm in pseudocode, **explain the meaning of your variables** (in English). WARNING: The bonus problems are underrated. Do the ordinary problems first. – This quiz contributes 6% to your course grade.

1. (6 points) A complete binary tree has n nodes. Determine the **exact** height of the tree (as a function of n). **Prove** your answer.

2. (9 points) Describe in elegant pseudocode the DECREASE-KEY(x, *newkey*) operation in a heap. (x is the name of a node; *newkey* is the value with which we replace key(x). We assume *newkey* $<$ key(x).)

3. (3+12 points) (a) State the Knapsack Problem. Conciseness and clarity are paramount. (b) Assume there are n items; the *values* are positive integers. Solve the problem in $O(nV)$ steps where V is the combined value of all items. Describe the algorithm in pseudocode. **Define your variables.** A clear mathematical definition of your variables will account for half the credit.

4. (a) (2 points) For two sequences of real numbers, $\{a_n\}$ and $\{b_n\}$, define the relation $a_n \sim b_n$ ("a_n is asymptotically equal to b_n") as in the handout.

   (b) (3 points) Explain why the statement "$(\forall n)(a_n \sim b_n)$" does not make sense. Your explanation should be short and to the point.

   (c) (3 points) For two sequences of real numbers, $\{a_n\}$ and $\{b_n\}$, define the relation $a_n \gtrsim b_n$ ("a_n is greater than or asymptotically equal to b_n").

   (d) (4 points) Give an example of two sequences of real numbers, $\{a_n\}$ and $\{b_n\}$, such that $a_n \gtrsim b_n$ but infinitely often $a_n < b_n$ and yet $a_n \not\sim b_n$.

5. (4+8+3B points) Consider the recurrent inequality $T(n) \leq 2T(n/2) + O(n)$. (a) Name two significantly different algorithms discussed in class of which the the cost analysis lead to this recurrence. (b) Assuming $T(1) = 0$, prove that $T(n) = O(n \log n)$. (Assume n is a power of 2.) (c) (BONUS) Prove the same conclusion assuming $T(1) = 1$.

6. (6+6B points) (Selection by rank) Given a list of n data (real numbers), we wish to find the k-th smallest among them (this is the element of rank k). We can only do comparisons with the data. (a) Describe a randomized algorithm which achieves this in an expected $O(n)$ number of comparisons. Describe the algorithm in elegant pseudocode. (b) (BONUS) Let $f(n)$ denote the expected number of comparisons performed by the algorithm. Prove that $f(n) = O(n)$. (Use the reverse of this sheet.)