Algorithms CMSC-37000 Second Quiz. February 5, 2009
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Name: ____

Show all your work. **Do not use book, notes, or scrap paper.** Write your answers in the space provided. You may **continue on the reverse.** When describing an algorithm in pseudocode, **explain the meaning of your variables** (in English). WARNING: The bonus problems are underrated. Do the ordinary problems first. – This quiz contributes 6% to your course grade.

1. (15 points, lose 6 per mistake) For each statement, decide whether or not it is a loop-invariant for BFS: (a) "Vertex #2 is black." (b) "Vertex #2 is white." (c) "Vertex #2 cannot change from black to white." Reason your "NO" answers.

2. (12 points) Consider the following statements regarding Dijkstra's algorithm. $Q_1 : (\forall u \in V)(u \in Q$ if and only if u is grey); $Q_2 : (\forall u \in V)($ if u is white then $c(u) = \infty)$; $Q_4 : (\forall v \in V)(c(v)$ is the minimum cost among all $s \to \cdots \to v$ paths passing through black vertices only).

   Prove that $Q_1 \& Q_2 \& Q_4$ is not a loop invariant. Draw and explain.

3. (10+4 points) (a) Evaluate the sum $\sum_{i=0}^{\infty} i/2^i$. Prove your answer. (b) State an algorithmic result which was proved in class using (a). Do not prove.

4. (12 points) Mario's Counting Service (MCS) stores a nonnegative integer x and upon request, increments it ($x \leftarrow x+1$). Mario stores x in binary; his cost of the increment is the number of bits changed. (So the cost of incrementing $x = 23$ is 4 because $23 = 10111_2$, $24 = 11000_2$.) The starting value of x is zero. Mario charges 2 units for any increment request. Can he afford this? (He just wants to break even.) What kind of "complexity analysis" does he need to perform? Perform this analysis.

5. (7 points) Given a graph by an adjacency array (array of adjacency lists), compute an adjacency array for the reverse graph (every edge reversed) in linear time. Write your algorithm in pseudocode. Do not refer to other algorithms.

6. (BONUS PROBLEM, 8+4 points) Let L be a list of vectors in a vector space and let $c : L \to \mathbb{R}$ be a "cost function." We wish to construct a minimum cost basis of L. (A basis of L is a maximal set of linearly independent vectors in L.) Use the greedy algorithm: sort the vectors by cost, say $c(v_1) \leq c(v_2) \leq \cdots \leq c(v_m)$, and, starting from the empty list B, for $j = 1$ to m, if v_j is not in the span of B then add v_j to B. (a) Prove that this algorithm yields an optimal basis B. Assume for simplicity that all vectors have different costs. (b) Show that the min-cost spanning-tree problem is a special case. Given a connected weighted graph, you need to associate a vector with each edge such that bases of this list of vectors correspond to spanning tree of the graph. (Write your solution on the reverse side.)