CMSC-37110 Discrete Mathematics
FINAL EXAM December 11, 2009

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This exam contributes 32% to your course grade.

Do **not** *use book, notes.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else.

1. (10+6+18+14+10B points) (a) State the Spectral Theorem. (b) Define the operator norm of an $n \times n$ real matrix B. (c) Prove: if A is a real symmetric $n \times n$ matrix then its operator norm is $\max |\lambda_i|$ where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A. Use the Spectral Theorem in the proof. (d) Prove: if B is an $n \times n$ real matrix (not necessarily symmetric) then the matrix $A = B^T B$ is symmetric and all its eigenvalues are nonnegative. (B^T is the transpose of B.) (e) (BONUS) Prove: if B is as in (d) then the operator norm of B is $\sqrt{\mu}$ where μ is the largest eigenvalue of $A = B^T B$.

2. (10+16+8B points) Let A be a real $n \times n$ matrix (not necessarily symmetric). (a) Prove that the left and right eigenvalues of A are the same. (Note that these are complex numbers.) (b) Let $x \in \mathbb{R}^n$ be a left eigenvector to eigenvalue λ and $y \in \mathbb{R}^n$ a right eigenvector to eigenvalue μ. Prove: if $\lambda \neq \mu$ then x and y are orthogonal. (c) (BONUS) Prove: if A has a right eigenbasis in $\mathbb{R}^n$ then A has a left eigenbasis in $\mathbb{R}^n$.

3. (20 points) Let T be the transition matrix of a finite Markov chain. Prove: all eigenvalues of T have absolute value ≤ 1. (Note again that these are complex numbers.)

4. (16+ 8B points) (a) Consider the recurrence $x_n = ax_{n-1} + bx_{n-2} + cx_{n-3}$. Find the values of the constants a, b, c such that $x_n = n^2$ satisfies the recurrence. (b) (BONUS) Generalize the answer from 3-term recurrence to k-term recurrence.

5. (10+10 points) (a) State Kuratowski's characterization of planar graphs. (b) Prove: if a connected graph has n vertices and $n + 2$ edges then it is planar.

6. (24 points) Prove: for all sufficiently large n, the probability that a random graph is planar is less than $2^{-0.49n^2}$.

7. (20 points) Prove: almost all graphs on n vertices have no clique (complete subgraph) of size $\geq 1 + 2\log_2 n$. (Hint: estimate the probability of cliques of size k. Do not substitute the value $1 + 2\log_2 n$ for k until the very end to avoid messy formulas.)

8. (4 + 14 + 6 + 8 + 8 points) We have n guests and n gift items. For each gift item, we draw a guest's name at random. The same name can be drawn multiple times. (a) What is the size of the sample space for this experiment? (b) A guest is unlucky if his/her name is never drawn. Let X be the number of unlucky guests. Determine $E(X)$. (c) Asymptotically evaluate your answer to (b). Give a very simple expression. (d) Let p_n denote the probability that $X = 0$ (nonoe of the guests is is unlucky)? Detemine p_n (give a simple closed-form expression). (e) True or false: $p_n < 1/3^n$ for all sufficiently large n, Prove your answer.

9. (8+14 points) (a) State the multinomial theorem (express $(x_1+\cdots+x_k)^n$ as a sum). (b) Count the terms in your expression. Your answer should be a very simple expression (a binomial coefficient).

10. (8 points) Let F_n denote the n-th Fibonacci number (starting with $F_0 = 0$, $F_1 = 1$). Prove: for all n, the numbers F_n and F_{n+2} are relatively prime.

11. (8+14 points) (a) Consider the infinite arithmetic progression $x_n = a + bn$ where a, b are positive integer constants. Prove: there exist two terms in the progression that are not relatively prime. (b) Prove: there exists a 100-term arithmetic progression $y_n = c + dn$ ($n = 0, 1, \ldots, 99$) where c, d are positive integer constants such that the 100 terms are pairwise relatively prime. Prove your answer.

12. (10+10 points) Let $a_n > 2$ and $b_n > 2$ be sequences of real numbers. Consider the following two statements: (1) $a_n = \Theta(b_n)$; (2) $\ln a_n \sim \ln b_n$. (a) Prove that (2) does not follow from (1). (b) Prove that if $a_n \to \infty$ then (2) follows from (1).

13. (24 points) Find an integer x between 1 and 30 such that for every integer $a \geq 0$ we have $a^x \equiv a^{7^{150}} \pmod{31}$. (The exponent is 7^{150}.) Do not use a calculator.

14. (2+8 points) Let X be a random nonnegative integer with 100 decimal digits; initial zeros are permitted. (Each of the 100 digits is chosen at random from $\{0, 1, \ldots, 9\}$.) (a) What is the size of the sample space of this experiment? (b) Estimate the probability that X is prime. Use the approximation $\ln 10 \approx 2.303$. Do not use a calculator. Your answer should be a simple fraction.

15. (BONUS 10B points) Let $n = pq$ where p, q are distinct primes. Prove that the following statement is false:
$(\forall a)(\text{if } \gcd(a, n) = 1 \text{ then } a^{n-1} \equiv 1 \pmod{n})$.