1. (6 points) Prove by induction on $k$: \( (\forall x)(\forall k \geq 1)(\text{if } x \text{ is odd then } x^{2^k} \equiv 1 \pmod{2^{k+2}}). \)

2. (2+3 points) (a) Define the greatest common divisors of a set $A$ of integers. (b) Prove: \( \gcd(ca, cb) = |c| \gcd(a, b) \). Do not use unique prime factorization. State and use a basic fact about $\gcd$ we proved in class. (Recall that the $\gcd$ notation refers to the nonnegative number among the greatest common divisors.)

3. (2+4 points) (a) Define the prime property. (b) Use the preceding problem to show that every prime number has the prime property.
4. (3 points) Assume $\gcd(x, y) = 1$. Prove: $\gcd(x + y, x - y) = 1$ or 2.

5. (2+4 points) True or false (circle one, prove). All quantifiers range over the integers.
   (a) $(\forall x)(\exists y)(\gcd(x, y) = x - y)$ T F
   (b) $(\forall x)(\exists y)(x^2 - y^2 \equiv 1 \pmod{7})$ T F

6. (3+4 points) Let $a = 5k + 1$ and $b = 3k - 2$. Prove: (a) There exist infinitely many values of $k$ such that $\gcd(a, b) \neq 1$. (b) If $\gcd(a, b) \neq 1$ then $\gcd(a, b) = 13$.

7. (2+2+3+6 points) Compute each multiplicative inverse or prove it does not exist; your answer $x$ (if exists) should be in the range $0 \leq x < m$ where $m$ is the modulus. $k, x$ are positive integers. (a) $7^{-1} \pmod{73}$ (b) $21^{-1} \pmod{91}$ (c) $k^{-1} \pmod{k^2 + k + 1}$ (d) BONUS: $(x + 1)^{-1} \pmod{x^2 + 1}$.

8. (8B points) BONUS. Let $r, s > 0$. Prove: $\gcd(2^s - 1, 2^t - 1) = 2^d - 1$ where $d = \gcd(r, s)$. 