CMSC-37110 Discrete Mathematics
SECOND QUIZ October 16, 2009

Name (print): ____________________________________

Do **not** *use book, notes, scratch paper.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else. **Write your solution in the space provided.** You may continue on the reverse. This exam contributes **6%** to your course grade.

1. (6+6 points) Prove your answers.
(a) Find x such that $16\mathbb{Z} \cap 28\mathbb{Z} = x\mathbb{Z}$ or prove that no such x exists.
(b) Find y such that $16\mathbb{Z} \cup 28\mathbb{Z} = y\mathbb{Z}$ or prove that no such y exists.

2. (10 points) Prove: $(\forall a)(a^{37} \equiv a \pmod{247}) \quad (247 = 13 \cdot 19)$

3. (4+1 points) (a) Assume $a, b > 0$. How many primes are there in the infinite arithmetic progression $an + b$, $n = 0, 1, 2, \dots$? Reason your answer. (b) Whose theorem?

4. (11 points) We want to distribute n identical chocolate bars to k children such that each of them gets at least 2 chocolate bars. How many ways can we do this? Your answer should be a very simple expression: a single binomial coefficient. Prove your answer.

5. (6+6 points) True or false: circle one, prove your answer.

   (a) $\binom{n}{2} \sim n^2/2$ **T** **F**

   (b) $2^{\binom{n}{2}} \sim 2^{n^2/2}$ **T** **F** (the binomial coefficient is in the exponent)

6. (5+5 points) Let E_n denote the number of even subsets of an n-set, and O_n the number of odd subsets of an n-set. For $n \geq 1$, prove: $E_n = O_n$. Give (a) an algebra proof; (b) a combinatorial (bijective) proof.

7. (6B points) BONUS. Let $S_4(n)$ be the number of those subsets of an n-set whose cardinality is divisible by 4. Give a closed-form expression (no summations or ellipses (dot-dot-dots)). Evaluate your answer for $n = 99$.

8. (6B points) BONUS. True or false? **T** **F** Prove your answer.

$$\left(1 + \frac{1}{n}\right)^{n^2} \sim \mathrm{e}^n.$$