Name (print): ____________________________

Do not use book, notes, scratch paper. Show all your work. If you are not sure of the meaning of a problem, ask the instructor. The bonus problems are underrated, do not work on them until you are done with everything else. Write your solution in the space provided. You may continue on the reverse. This exam contributes 6\% to your course grade.

1. (5+5 points) Let \( \{a_n\} \) be a sequence of positive numbers. Consider the following two statements. (A) \( a_n = O(2^{b_n}) \); (B) \( a_n = 2^{O(b_n)} \). (Both statements involve the big-Oh notation.) (a) Prove: (A) \( \not\Rightarrow \) (B).
   (b) Prove: if we additionally assume \( a_n, b_n > 0.01 \) then (A) \( \Rightarrow \) (B).

2. (2+4+12+2+15B points) Let \( w \) be a random string of length \( n \geq 3 \) over the alphabet \( \{A, B\} \) (uniform distribution). Let \( X \) denote the number of occurrences of \( ABA \) as a contiguous substring in \( w \). (For instance, if \( w = ABABABB \) then \( X = 2 \).) (a) What is the size of the sample space for this experiment? (b) Compute \( E(X) \). (c) Compute \( \text{Var}(X) \).
   (d) Asymptotically evaluate \( \text{Var}(X) \).
   (e) (BONUS) Prove: there exist positive constants \( c, d \) such that
   \[ (\forall n)(\forall \epsilon > 0)(P(|X - E(X)| > \epsilon n) < c \exp(-d\epsilon^2 n)). \]
   (Notation: \( \exp(x) = e^x \).)
3. (4+3+8 points) (a) For which polynomials $f \in \mathbb{Q}[x]$ is it true that $(\forall g \in \mathbb{Q}[x])(f \mid g)$? (b) Define: when is a polynomial $f \in \mathbb{Q}[x]$ irreducible over $\mathbb{Q}$? (c) Prove: if $f \in \mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$ then $f$ has no multiple complex roots. Do not use any homework exercises. (If you must, include the proof.)

4. (3+4+8 points) Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a monic polynomial ($a_n = 1$) over $\mathbb{C}$. Let $f(x) = (x - \alpha_1) \cdots (x - \alpha_n)$ ($\alpha_i \in \mathbb{C}$). (a) Express $a_{n-1}$ and (b) $a_{n-2}$ through $\alpha_1, \ldots, \alpha_n$. (c) Let $S_2 = \sum_{i=1}^{n} \alpha_i^2$. Express $S_2$ through the coefficients of $f$ (i.e., through $a_{n-1}, a_{n-2}$, etc.) Give a very simple expression. No proof necessary for (a) and (b); prove (c).

5. (BONUS, 12B points) Prove: for all sufficiently large $n$, the probability that a random graph on $n$ vertices is bipartite is less than $2^{-0.24n^2}$. 