CMSC-37110 Discrete Mathematics
FOURTH QUIZ December 2, 2009

Name (print): ____________________

Do **not** *use book, notes, scratch paper.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else. Begin your solution in the space provided; **continue on the reverse.**
This exam contributes 6% to your course grade.

1. (12 points) Let F be a field, $A \in M_n(F)$, and assume $v_1, \ldots, v_k$ are eigenvectors of A corresponding to distinct eigenvalues. Prove that $v_1, \ldots, v_k$ are linearly independent.

2. (6+4+4 points) (a) Suppose the eigenvalues of a matrix $A \in M_3(\mathbb{R})$ are 2, 3, 5. Calculate the trace and the determinant of A. (b) Prove that A is diagonalizable. (What does this mean?) (c) Construct such a matrix A with at most 3 entries equal to zero.

3. (3+2+10 points) (a) Define what it means for a polynomial $f \in \mathbb{Q}[x]$ to be irreducible over $\mathbb{Q}$. (b) Let $A \in M_n(\mathbb{Z})$ be an $n \times n$ matrix with characteristic polynomial $f_A(t)$. Prove that the coefficients of $f_A(t)$ are integers. (c) Assume $f_A(t)$ is irreducible over $\mathbb{Q}$. Prove that A has an eigenbasis over $\mathbb{C}$.

4. (4+3 points) Consider the set R of polynomials of degree ≤ 5 over $\mathbb{F}_p$ that are divisible by the polynomial $g(x) = x^2 + 2x - 1$. This is a subspace of $\mathbb{F}_p[x]$. (a) Determine $\dim R$; find a basis of R. (b) If we pick a random polynomial f of degree ≤ 5, what is the probability that $f \in R$? What is the size of the sample space for this experiment?

5. (12 points) Prove: for all sufficiently large n, the probability that a random graph on n vertices is planar is less than $2^{-0.49n^2}$.

6. (BONUS, 8B points) Let $G = ([n], E)$ be a graph with m edges. Consider the set $P(G) = \{x^i - x^j : i > j, \{i,j\} \in E\}$ consisting of m polynomials over the field F. Prove that $P(G)$ is linearly independent if and only if G is a forest (i. e., G has no cycles).

7. (BONUS, 15B points) Assume all eigenvalues λ_i of $A \in M_n(\mathbb{C})$ satisfy $|\lambda_i| < 1$. Prove: $\lim_{k\to\infty} A^k = 0$. Use the fact that over $\mathbb{C}$, every matrix is similar to a triangular matrix. Do not use the Jordan normal form.