

4. (4+3 points) Consider the set R of polynomials of degree ≤ 5 over \mathbb{F}_p that are divisible by the polynomial $g(x) = x^2 + 2x - 1$. This is a subspace of $\mathbb{F}_p[x]$. (a) Determine $\dim R$; find a basis of R . (b) If we pick a random polynomial f of degree ≤ 5 , what is the probability that $f \in R$? What is the size of the sample space for this experiment?
5. (12 points) Prove: for all sufficiently large n , the probability that a random graph on n vertices is planar is less than $2^{-0.49n^2}$.
6. (BONUS, 8B points) Let $G = ([n], E)$ be a graph with m edges. Consider the set $P(G) = \{x^i - x^j : i > j, \{i, j\} \in E\}$ consisting of m polynomials over the field F . Prove that $P(G)$ is linearly independent if and only if G is a forest (i. e., G has no cycles).
7. (BONUS, 15B points) Assume all eigenvalues λ_i of $A \in M_n(\mathbb{C})$ satisfy $|\lambda_i| < 1$. Prove: $\lim_{k \rightarrow \infty} A^k = 0$. Use the fact that over \mathbb{C} , every matrix is similar to a triangular matrix. Do not use the Jordan normal form.