1. (6+10 points) A divide-and-conquer algorithm reduces an instance of size $n$ to four instances of size $n/3$ each. The cost of the reduction is $O(n)$. Let $T(n)$ denote the cost of the algorithm on the worst instance of size $n$. (a) State the recurrent inequality for $T(n)$ that follows from such a reduction. (b) Use the method of reverse inequalities to prove that $T(n) = O(n^\beta)$; determine the best possible $\beta$ achievable based on the information given. (Assume $n = 3^k$.)

2. (6 points) Give a simple description of all sequences $\{a_n\}$ that are asymptotically equal to the 0, 0, . . . ("all-zero") sequence. Do not prove.

3. (12 points) Describe in elegant pseudocode the DECREASE-KEY($x$, $newkey$) operation in a heap. ($x$ is the name of a node; $newkey$ is the value with which we replace $key(x)$. We assume $newkey < key(x)$.)
4. (12 points) Given a sorted array \( A[1..n] \) of \( n \) real numbers \( A[1] \leq A[2] \leq \cdots \leq A[n] \) and a real number \( x \), decide whether or not \( x \) is in the array. Use the minimum possible number of comparisons. (Comparisons are binary; they answer questions of the form “is \( y \leq z \)?”) Write your algorithm in pseudocode. State the name of the algorithm. Do not assume that \( n \) is a power of 2; be careful about rounding. Do not analyze the algorithm.

5. (14 points) Let \( a_n, b_n \) be sequences of reals. Recall that we say that \( a_n \gtrsim b_n \) if \( a_n \sim \max\{a_n, b_n\} \). Assume \( a_n \gtrsim b_n \). Prove that it does not follow that \( 2^{a_n} = \Omega(2^{b_n}) \).

6. (BONUS PROBLEM, 8 points) (How helpful is a heap?) Andrea designed an algorithm that takes as input \( n \) data (real numbers) arranged in a heap, performs at most \( t(n) \) comparisons on the data, and returns the sorted list of the data. Prove that \( t(n) \gtrsim n \log_2 n \). Warning: this is NOT a question about Heapsort! Andrea’s algorithm has random access to the data; it can ignore the heap or rely on comparisons implied by the heap as needed. The only type of operations Andrea’s algorithm is permitted to do on the data are comparisons. Bookkeeping is free. (This includes copying, comparing, and doing arithmetic with addresses, recording the outcomes of previous comparisons, etc.)