Combinatorics Math284/CMSC274/372
Take-home Exam. Due June 2, 2010
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Problems 1-9 posted May 26, 10:20pm, the rest May 28, 6am.

Show all your work. **Prove everything you use unless it was proved in class.** You may use sources (text, web). Do not copy; understand, and reproduce in your own words. Name your sources. **Do NOT collaborate. Explain the meaning of your variables** (in English). Make your proofs short and clear. **Elegance counts.** WARNING: The bonus problems are underrated. Do the ordinary problems first. – This exam contributes 25% to your course grade.

1. (16 points) Prove: the probability that a random graph does not contain a clique of size 100 is $< C^{-n^2}$ for some constant $C > 1$ and all sufficiently large n. (Edges are chosen with probability 1/2.)

2. (10 points) Prove: for all $c > 0$ there exists $d > 0$ such that if a graph G on n vertices does not contain a clique of size $\lceil c \log n \rceil$ then it contains an independent set of size $\lceil d \log n \rceil$.

3. (15 points) Prove: for almost all graphs G we have $\chi(G) = \Theta(n/\log n)$.

4. (8 points) Let $L(n)$ denote the number of Latin squares on a given set of n points. Let $N(n)$ denote the number of non-isomorphic Latin Squares of order n. Prove: $\log L(n) \sim \log N(n)$.

5. (3 points) Define what it means for n random variables $X_1, \dots, X_n$ to be independent.

6. (2+4+12+8 points) Consider the Erdős-Rényi random graph $G_{n,p}$. This graph has n vertices and each of the $\binom{n}{2}$ pairs is chosen with probability p to be an edge (independently). **(a)** What is the size of the sample space for this experiment? **(b)** Let X denote the number of 4-cliques (copies of K_4) in $G_{n,p}$. Determine $E(X)$. Prove. Define your variables. **(c)** Determine $\mathrm{Var}(X)$. Your answer should be a simple closed-form expression (no summation/product signs, no dot-dot-dots). Prove your formula. **(d)** Prove: if p_n is a sequence of numbers, $0 < p_n < 1$, and $\lim_{n\to\infty} n^{2/3} p_n = \infty$, then with high probability (w.h.p.), G_{n,p_n} contains a 4-clique. ("W.h.p." means the probability approaches 1 as $n \to \infty$.) Name the method used.

7. (8 points) Prove: for every n, there exists a graph with n vertices and $\Omega(n^{3/2})$ edges which does not contain a 4-cycle. Estimate the constant implied by the Ω notation for large n. (The bigger, the better.) Prove.

8. (8 points) Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix of a graph. Prove: $\sum \lambda_i^2 = 2m$ where m is the number of edges.

9. (8 points) Let $S(n,4) = \sum_{k=0}^{\infty} \binom{n}{4k}$. Find all values of n such that $S(n,4) = 2^{n-2}$.

10. (6+6+6+6 points) An r-matching of a graph is a set of r disjoint edges. Count the r-matchings of (a) the complete bipartite graph $K_{s,t}$; (b) the complete graph K_n; (c) the path P_n of length $n-1$ (it has n vertices); (d) the cycle C_n of length n. Your answer to each question should be a simple closed-form expression.

11. (20 points) The *matchings polynomial* of the graph G is defined as

$$m(G,x) = \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r p(G,r) x^{n-2r}, \tag{1}$$

where $p(G,r)$ is the number of r-matchings of G. Prove: $m(C_n, x) = 2T_n(x/2)$ where $T_n(x)$ is the Chebyshev polynomial of the first kind. ($T_n(x)$ is defined by the identity $T_n(\cos\theta) = \cos(n\theta)$.)

12. (24 points) Let $f_G(x)$ denote the characteristic polynomial of the adjacency matrix A of the graph G, i. e., $f_G(x) = \det(xI - A)$. Prove: if G is a tree then $f_G(x) = m(G,x)$.

13. (12+6+6 points) (a) The projective plane $\mathcal{P}' = (P', L', I')$ is a *subplane* of the projective plane $\mathcal{P} = (P, L, I)$ if $P' \subseteq P$, $L' \subseteq L$, and $I' = I \cap (P' \times L')$. A *proper* subplane is a subplane that is not the entire plane. Prove: if a projective plane of order m is a proper subplane of a projective plane of order n then $m \leq \sqrt{n}$. (Recall that the *order* of a projective plane is one less than the number of points on a line.) (b) We say that a subset $S \subseteq P$ generates the projective plane $\mathcal{P} = (P, L, I)$ if S is not contained in any proper subplane. Prove: a projective plane of order n has a set of $\leq 3 + \log_2 \log_2 n$ generators. (c) Prove that $|\operatorname{Aut}(\mathcal{P})| \leq (n+1)^{6+2\log_2\log_2 n}$.

14. (6+4 points) Let A be a $(0,1)$-matrix (every entry is 0 or 1). (a) Prove: if the columns of A are linearly independent over the field of order 2 then they are linearly independent over the reals. (b) Prove that the converse is false. Make your counterexample as small as possible.

15. (20 points) Let $X_1, \ldots, X_k$ be pairwise independent non-constant random variables over a probability space of size n. Prove: $k \leq n$.

16. (8 points) Let $S(n)$ denote the number of Steiner Triple Systems (STS) on the point set $[n]$. Let $L(n)$ denote the number of $n \times n$ Latin Squares. Prove: $S(3n) \geq (S(n))^3 L(n)$.

17. (18 points) Let $v_1, \dots, v_k$ be vectors in $\mathbb{R}^2$ of norm ≥ 1. Let $r \in \mathbb{R}^2$. Let I be a random subset of $[k]$. Prove: $P(||\sum_{i \in I} v_i - r|| < 100) = O(1/\sqrt{k})$.

18. (BONUS: 6B+6B points)
    (a) Prove: if the graph G has no paths of length k then G is k-colorable ($\chi(G) \leq k$).
    (b) Prove: for every k there exists c_k such that if the graph G does not contain k-cycles then $\chi(G) \leq c_k\sqrt{n}$.

19. (BONUS: 6B points) Let f be a nonzero polynomial. Prove: there exists a nonzero polynomial g such that only prime numbers occur as exponents (with nonzero coefficient) in the polynomial fg. (This is true over any field.)

20. (BONUS: 8B points) Prove: the probability that a random graph is regular is $< n^{-cn}$ for some positive constant c. ($n \geq 3$ is the number of vertices.)

21. (BONUS: 10B points) For a graph G, let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denote the eigenvalues of the adjacency matrix. Let $\mu = \max\{\lambda_2, |\lambda_n|\}$. Prove: for almost all graphs, $\mu = o(n)$.