Name: 

Show all your work. Do not use book, notes, or scrap paper. Write your answers in the space provided and continue on the reverse if necessary. Explain the meaning of your variables (in English). Notation: $[n] = \{1, 2, \ldots, n\}$. WARNING: The bonus problems are underrated. Do the ordinary problems first. – This quiz contributes 6% to your course grade.

1. (20 points) (BLYM inequality) Recall that a Sperner family is an antichain of sets. Prove: if $A_1, \ldots, A_m \subseteq [n]$ is a Sperner family then
$$\sum_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}} \leq 1.$$ 
Define your variables! (Reproduce the proof from class.)

2. (10 points) Count the 4-cycles in the complete bipartite graph $K_{r,s}$. Your answer should be a simple closed-form expression (no summation or dot-dot-dots). Just give the formula, do not prove.

3. (5+5 points) (a) Define when do we say that two sequences $\{a_n\}$ and $\{b_n\}$ are asymptotically equal ($a_n \sim b_n$). (b) State Stirling’s formula. Use the $\sim$ notation.
4. (12 points) True or false? Circle one. If “false,” draw a small counterexample. $G$ is a graph with $n$ vertices and $m$ edges; $\chi(G)$ is its chromatic number; $\alpha(G)$ is the size of the largest independent set.

(a) If $\chi(G) \geq 4$ then $G$ contains $K_4$. $\text{T F}$
(b) If $\chi(G) \leq 4$ then $\alpha(G) \geq n/4$. $\text{T F}$
(c) If $\alpha(G) \geq n/4$ then $\chi(G) \leq 4$. $\text{T F}$
(d) $m \leq n(n-1)/2$. $\text{T F}$
(e) If $m \geq n(n-1)/3$ then $G$ is connected. $\text{T F}$

5. (8 + 8B points) In “Reverse Oddtown,” we assume each $|A_i|$ is even and each $|A_i \cap A_j|$ is odd. Prove that in this case (a) $m \leq n + 1$ (b) (BONUS) $m \leq n$.

6. (BONUS: 6B+6B points)

(a) Prove: if the graph $G$ has no paths of length $k$ then $G$ is $k$-colorable ($\chi(G) \leq k$).
(b) Prove: for every $k$ there exists $c_k$ such that if the graph $G$ does not contain $k$-cycles then $\chi(G) \leq c_k \sqrt{n}$.