Combinatorics Math 284/CMSC-372 First Quiz. April 9, 2010
Instructor: László Babai

Name: ______________________________

Show all your work. **Do not use book, notes, or scrap paper.** Write your answers in the space provided and **continue on the reverse** if necessary. **Explain the meaning of your variables** (in English). Notation: $[n] = \{1, 2, \dots, n\}$. WARNING: The bonus problems are underrated. Do the ordinary problems first. – This quiz contributes 6% to your course grade.

1. (20 points) (BLYM inequality) Recall that a *Sperner family* is an antichain of sets. Prove: if $A_1, \dots, A_m \subseteq [n]$ is a Sperner family then

$$\sum_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}} \le 1.$$

Define your variables! (Reproduce the proof from class.)

2. (10 points) Count the 4-cycles in the complete bipartite graph $K_{r,s}$. Your answer should be a simple closed-form expression (no summation or dot-dot-dots). Just give the formula, do not prove.

3. (5+5 points) (a) Define when do we say that two sequences $\{a_n\}$ and $\{b_n\}$ are asymptotically equal ($a_n \sim b_n$). (b) State Stirling's formula. Use the $\sim$ notation.

4. (12 points) True or false? Circle one. If "false," draw a small counterexample. G is a graph with n vertices and m edges; $\chi(G)$ is its chromatic number; $\alpha(G)$ is the size of the largest independent set.

   (a) If $\chi(G) \geq 4$ then G contains K_4. **T** **F**

   (b) If $\chi(G) \leq 4$ then $\alpha(G) \geq n/4$. **T** **F**

   (c) If $\alpha(G) \geq n/4$ then $\chi(G) \leq 4$. **T** **F**

   (d) $m \leq n(n-1)/2$. **T** **F**

   (e) If $m \geq n(n-1)/3$ then G is connected. **T** **F**

5. (8 + 8B points) In "Reverse Oddtown," we assume each $|A_i|$ is even and each $|A_i \cap A_j|$ is odd. Prove that in this case (a) $m \leq n+1$ (b) (BONUS) $m \leq n$.

6. (BONUS: 6B+6B points)

   (a) Prove: if the graph G has no paths of length k then G is k-colorable ($\chi(G) \leq k$).

   (b) Prove: for every k there exists c_k such that if the graph G does not contain k-cycles then $\chi(G) \leq c_k\sqrt{n}$.