Combinatorics Math 284/CMSC-372 Second Quiz. April 16, 2010
Instructor: László Babai

Name: ____

Show all your work. **Do not use book, notes, or scrap paper.** Write your answers in the space provided and **continue on the reverse** if necessary. **Explain the meaning of your variables** (in English). WARNING: The bonus problems are underrated. Do the ordinary problems first. – This quiz contributes 6% to your course grade.

Notation: $[n] = \{1, 2, \ldots, n\}$. In the problems below, $\mathcal{F} = \{A_1, \ldots, A_m\}$ denotes a set system.

1. (3+14 points) (a) Define what it means that $\mathcal{F}$ is 2-colorable. (b) (Erdős) Prove: if $\mathcal{F}$ is r-uniform and $m \le 2^{r-1}$ then $\mathcal{F}$ is 2-colorable.

2. (BONUS: 5B points) Let $S(n, 4) = \sum_{k=0}^{\infty} \binom{n}{4k}$. Find all values of n such that $S(n, 4) = 2^{n-2}$.

3. (4+3+6 points) (a) Define $\tau(\mathcal{F})$ (the covering number) and $\nu(\mathcal{F})$ (the matching number). (b) State the inequality between τ and ν that holds for all $\mathcal{F}$. (c) Recall Dénes Kőnig's Theorem that for bipartite graphs, $\tau = \nu$. Draw a small graph that satifies $\tau = \nu$ even though it is <u>not</u> bipartite.

4. (4+10 points) Recall that a SDR (system of distinct representatives) is a set of distinct elements x_i such that $x_i \in A_i$ $(i = 1, \dots, m)$. (a) State Philip Hall's "SDR Theorem," which gives a complete set of obstacles to having an SDR. Your statement should begin with "$\mathcal{F}$ has a SDR if and only if ..." (b) Deduce the SDR Theorem from Kőnig's $\tau = \nu$ theorem.

5. (16 points) Recall that an $n \times n$ matrix $A = (a_{ij})$ is *doubly stochastic* if $(\forall i, j)(a_{ij} \geq 0)$ and the sum of every row and the sum of every column is 1. Recall that the permanent of a matrix $A = (a_{ij})$ is the sum $\mathrm{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}$ (S_n is the group of all permutations of $[n]$). Prove: if A is doubly stochastic then $\mathrm{per}(A) \neq 0$. (Hint: SDR.)

6. (BONUS: 6B points) In "Reverse Oddtown," we assume each $|A_i|$ is even and each $|A_i \cap A_j|$ is odd. Prove that in this case $m \leq n$.