Combinatorics Math284/CMSC274/372 Third Quiz. May 5, 2010
Instructor: László Babai

Name: ______________________________

Show all your work. **Do not use book, notes, or scrap paper.** Write your answers in the space provided and **continue on the reverse** if necessary. **Explain the meaning of your variables** (in English). WARNING: The bonus problems are underrated. Do the ordinary problems first. – This quiz contributes 6% to your course grade.

1. (6 points) Prove: the adjacency matrix of a nonempty graph is indefinite (it has at least one positive and at least one negative eigenvalue).

2. (12 points) Let λ be the largest eigenvalue of the adjacency matrix of the graph G, and let $\overline{d}$ denote the average degree of the graph. Prove: $\lambda \geq \overline{d}$. (Hint: Rayleigh principle: the largest eigenvalue of a symmetric matrix is the maximum of its Rayleigh quotient.)

3. (16 points) Prove: $\Theta(G) \leq \chi(\overline{G})$. Here $\Theta(G)$ denotes the Shannon capacity of the graph G and $\overline{G}$ denotes the complement of G.

4. (10 points) Prove: if the graph G is self-complementary then $\Theta(G) \geq \sqrt{n}$. (Here n is the number of vertices; the graph G is *self-complementary* if $G \cong \overline{G}$.)

5. (16 points) Let $\mathcal{H}$ be an r-uniform hypergraph with n vertices and m edges. Prove: $\tau(\mathcal{H}) \leq \lceil (n/r) \ln m \rceil$.

6. (BONUS 6 points) Describe the common (complex) eigenbasis to all $n \times n$ circulant matrices. Recall that each row of a circulant matrix is obtained by a cyclic shift to the right from the preceding row. Example: this is the generic 4×4 circulant matrix:

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_3 & a_0 & a_1 & a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_1 & a_2 & a_3 & a_0 \end{pmatrix}$$

7. (BONUS 4 points) Let A be a positive definite $n \times n$ symmetric real matrix. Prove: there exists a positive definite real symmetric matrix B such that $A = B^2$.