Combinatorics Math284/CMSC274/372 Fourth Quiz. May 14, 2010
Instructor: László Babai

Name: ______________________________________

Show all your work. **Do not use book, notes, or scrap paper.** Write your answers in the space provided and **continue on the reverse** if necessary. **Explain the meaning of your variables** (in English). WARNING: The bonus problems are underrated. Do the ordinary problems first. – This quiz contributes 6% to your course grade.

1. (4+8+8 points) Let $A = (a_{ij})$ be random $n \times n$ $(0,1)$-matrix: each entry is 0 or 1, chosen at random by independent unbiased coin flips. (a) What is the size of the sample space? (b) Calculate the expected value of $\text{Per}(A)$ and (c) $\text{Det}(A)$. Your answers should be very simple formulas (closed form expressions, no summation or dot-dot-dot). Prove your answers to (b) and (c).

2. (8 points) Find the eigenvalues and an eigenbasis of the J matrix ($n \times n$ all-ones matrix: all entries are 1).

3. (3+9+12 points) Let X be a projective plane or order n (meaning: the lines have $n+1$ points). Let A be the incidence matrix of X (i. e., the (0,1)-matrix of which the rows are the incidence vectors of the lines). (a) State the dimensions of this matrix. (b) Compute the matrix $A^T A$. (c) Find $|\det(A)|$.

4. (8 points) Recall that an *orthonormal reprensentation* of a graph $G = ([n], E)$ (according to László Lovász) assigns unit vectors $v_1, \dots, v_n$ to the vertices such that if i and j are distinct non-adjacent vertices then v_i and v_j are perpendicular. Determine, which graphs have an orthonormal representation in $\mathbb{R}^2$. Your answer should be a very simple characterization in familiar terms of graph theory. Do not prove.

5. (BONUS, 6B points) Let p_1, p_2, p_3, p_4 be four points in the Galois plane $\mathrm{PG}(2, q)$ in general position (no three on a line). Let r_1, r_2, r_3, r_4 be another four points in general position. A "collineation" is a permutation of the points of the plane that preserves lines. Prove: there exists a collineation that moves p_i to r_i for all i.

6. (BONUS, 6B points) Prove: the automorphism group of the Petersen graph is isomorphic to S_5.