Combinatorics Math284/CMSC274/372 Fifth Quiz. May 26, 2010
Instructor: László Babai

Name: ____

Show all your work. **Do not use book, notes, or scrap paper.** Write your answers in the space provided and **<u>continue on the reverse</u>** if necessary. **Explain the meaning of your variables** (in English). WARNING: The bonus problems are underrated. Do the ordinary problems first. – This quiz contributes 6% to your course grade.

1. (8 points) Let $S(n)$ denote the number of Steiner Triple Systems (STS) on the point set $[n]$. Let $L(n)$ denote the number of $n \times n$ Latin Squares. Prove: $S(3n) \geq (S(n))^3 L(n)$.

2. (7 points) Define what it means for n random variables $X_1, \dots, X_n$ to be independent.

3. (5 points) We roll n dice. Let X be the product of the n numbers that come up. Determine $E(X)$. State, do not prove.

4. (2+3+8+12 points) Consider the Erdős-Rényi random graph $G_{n,p}$. This graph has n vertices and each of the $\binom{n}{2}$ pairs is chosen with probability p to be an edge (independently). **(a)** What is the size of

the sample space for this experiment? **(b)** Let X denote the number of triangles in $G_{n,p}$. Determine $E(X)$. (State, do not prove.) **(c)** Determine $\mathrm{Var}(X)$. Your answer should be a closed-form expression (no summation/product signs, no dot-dot-dots). Prove your formula. **(d)** Prove: if p_n is a sequence of numbers, $0 < p_n < 1$, and $\lim_{n\to\infty} np_n = \infty$, then with high probability (w.h.p.), G_{n,p_n} contains a triangle. ("W.h.p." means the probability approaches 1 as $n \to \infty$.) Name the method used.

5. (15 points) Prove: for every n, there exists a graph with n vertices and $\Omega(n^{3/2})$ edges which does not contain a 4-cycle. Estimate the constant implied by the Ω notation for large n. (The bigger, the better.) Prove.

6. (BONUS: 8B points) Prove: an STS of order n has at most $n^{\log_2(n+1)}$ automorphisms. (Prove your lemmas; do not use results that were not proven in class.)

7. (BONUS: 8B points) Let $X_1, \dots, X_k$ be pairwise independent nonconstant random variables over a probability space of size n. Prove: $k \le n$.