CMSC-37110 Discrete Mathematics
FINAL EXAM December 8, 2010

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This exam contributes 30% to your course grade.

Do **not** *use book, notes.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else.

1. (15 points) Suppose the nonzero vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ are orthogonal, i. e., for $i \neq j$ we have $v_i^T v_j = 0$. Prove that $v_1, \ldots, v_k$ are linearly independent.

2. (8 points) Draw a simple connected planar graph in two ways in the plane so that the resulting dual graphs are not isomorphic. Indicate a simple reason why they are not isomorphic. Make your graph as small as possible (minimum number of edges). Do not prove minimality. To make your drawings clear and unambiguous, do not draw the duals. (You may draw the graph combined with the dual separately.)

3. (3+6+12+3+8B points)

   (a) Define the norm of a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ under the standard dot product.

   (b) Define the operator norm $\|A\|$ of an $n \times n$ real matrix A.

   (c) Prove: if $B = (\beta_{i,j})$ is an $n \times n$ real matrix then $|\beta_{i,j}| \leq \|B\|$.

   (d) Prove: if B is an $n \times n$ matrix then $B^T B$ is symmetric.

   (e) (BONUS) Prove that the operator norm of a (not necessarily symmetric) $n \times n$ real matrix B is $\sqrt{\lambda}$ where λ is the largest eigenvalue of $B^T B$.

4. (8+8 points) Consider the following two congruences.

   ($*$) $21x \equiv 35 \pmod{60}$

   ($**$) $12x \equiv 20 \pmod{240}$

   Decide whether or not each of the following holds:

   (a) $(\forall x)((*) \Rightarrow (**))$;

   (b) $(\forall x)((**) \Rightarrow (*))$.

   Prove your answers.

5. (15 points) Let p be a prime number and b an arbitrary integer. Prove: if $b^{p+1} \equiv 1 \pmod{p}$ then $b \equiv \pm 1 \pmod{p}$.

6. (3+3+8 points)

   (a) Define the big-Oh relation: for two sequences $\{a_n\}$ and $\{b_n\}$, we say that $a_n = O(b_n)$ if ... – finish the sentence. Give a properly quantified formula, no English words.

   (b) Define the big-Omega relation: for two sequences $\{a_n\}$ and $\{b_n\}$, we say that $a_n = \Omega(b_n)$ if ... – finish the sentence. Give a properly quantified formula, no English words.

   (c) Give an example of two sequences, $\{a_n\}$ and $\{b_n\}$, such that both sequences go to infinity, but $a_n \neq O(b_n)$ and $a_n \neq \Omega(b_n)$. Just state the sequences; you do not need to prove the correctness of your example (as long as it is correct).

7. (8+18 points) (a) State Kuratowski's characterization of planar graphs. Define the concept involved (topological ...). (b) Prove: if a connected graph has n vertices and $n+2$ edges then it is planar.

8. (20 points) Consider a deck of $4n$ cards, divided into fours suits ($\spadesuit$, $\heartsuit$, $\diamondsuit$, $\clubsuit$) of n cards each. What is the probability that a hand of k cards includes at least one card of each of the four suits? Give a simple closed-form expression. Name the method used.

9. (3 + 12 + 4 + 6 + 6 points) We have n guests and n gift items. For each gift item, we draw a guest's name at random. The same name can be drawn multiple times. (a) What is the size of the sample space for this experiment? (b) A guest is unlucky if his/her name is never drawn. Let X be the number of unlucky guests. Determine $E(X)$. (c) Asymptotically evaluate your answer to (b). Give a very simple expression. (d) Let p_n denote the probability that $X = 0$ (all guests are lucky). Determine p_n (give a simple closed-form expression). (e) True or false: $p_n < 1/2.7^n$ for all sufficiently large n. Prove your answer. (Note: $\mathrm{e} = 2.718\ldots$.)

10. (8+14 points) (a) State the multinomial theorem (express $(x_1 + \cdots + x_k)^n$ as a sum). Evaluate the multinomial coefficients appearing in your expression. (b) Count the terms of the sum in your expression. Your answer should be a very simple expression (a binomial coefficient). Prove your answer.

11. (3+8+6+8+6 points) A *tournament* is a directed graph with the property that for every pair x, y of distinct vertices, exactly one of the directed edges $x \to y$ and $y \to x$ is present. (In other words, a tournament is an "orientation of the complete graph.") Loops are not permitted. To obtain a *random tournament* on n vertices, we take K_n and flip a coin for each pair of vertices to decide which way to orient the edge between them. Let $T = (V, E)$ be a random tournament on n vertices.

(a) State the size of the sample space for this experiment.

(b) Let X denote the number of directed cycles of length 3 in T. Calculate $E(X)$. Your answer should be a simple formula. Prove your answer. Give a careful and clear definition of the random variables used. This definition accounts for half the credit.

(c) Let A and B be two distinct (but not necessarily disjoint!) subsets of V; let $|A| = |B| = 3$. Let $C(A)$ denote the event that the subdigraph of T induced on A is a directed cycle (of length 3); the event $C(B)$ is defined analogously. Prove that $C(A)$ and $C(B)$ are independent.

(d) Determine the variance of X. Prove your answer.

(e) Prove the Weak Law of Large Numbers for X. In other words, for $\epsilon > 0$, let $p_n(\epsilon)$ denote the probability that $|X - E(X)| \geq \epsilon E(X)$. Prove: $(\forall \epsilon > 0)(\lim_{n\to\infty} p_n(\epsilon) = 0)$.

12. (8+8+8 points) Recall that the generating function of a sequence $\{a_n\}$ is the power series $\sum_{n=0}^{\infty} a_n x^n$. Give a closed-form expression for the generating function of each of the following sequences:

(a) $a_n = n^2$;

(b) $b_n = b_{n-1} + 6b_{n-2}$ $(n \geq 2)$ and $b_0 = 3$, $b_1 = 1$;

(c) $c_n = 1$ if $n \equiv 4 \pmod 7$ and $c_n = 0$ otherwise $(n = 0, 1, \dots)$.

13. (3+10+18 points)

(a) Define what is a stationary distribution for a finite Markov Chain with n states.

(b) Find a finite Markov Chain which has a unique stationary distribution but this distribution is not everywhere positive (not every vertex has a positive stationary probability). Describe your Markov Chain both in diagram and by the transition matrix. Make your Markov Chain as small as possible. Prove your answer. (In particular, prove the uniqueness of the stationary distribution of your Markov Chain).

(c) Prove: if the Markov Chain is irreducible (i. e., the corresponding digraph is strongly connected) then its stationary distribution is all-positive (every state has a positive stationary probability).

14. (5+18 points) (a) Define the statement $n \to (k, \ell)$ (Erdős - Rado arrow notation in Ramsey theory). (b) Prove $10 \to (4, 3)$. You may use without proof that $6 \to (3, 3)$.

15. (BONUS 8B points) Let G be a directed graph such that the outdegree of every vertex is $\leq k$. Prove that G is $(2k + 1)$-colorable. (A legal

coloring of a directed graph is just a legal coloring of the corresponding undirected graph obtained by ignoring orientation.)

16. (BONUS 10B points) Let $A = (a_{i,j})$ be a symmetric $n \times n$ real matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j}^2 = \sum_{i=1}^{n} \lambda_i^2.$$

17. (BONUS 8B points) Prove: there are infinitely many values of k such that $2^k + 1$ and $3^k + 1$ are not relatively prime.