1. (8+8+8 points) Let $X, Y, Z$ be random variables over a finite probability space. (a) Define what it means that $X, Y, Z$ are independent. Do not include superfluous conditions. (b) Prove: if $X, Y, Z$ are independent then they are pairwise independent. (c) Prove that the following statement is false: if $X, Y, Z$ are pairwise independent then they are independent. Make your sample space small.

2. (2+9 points) We roll $n$ (6-sided) dice. Let $X$ denote the product of the $n$ numbers shown. (a) What is the size of the sample space? (b) Determine $E(X)$. Your answer should be a very simple formula.

3. (1+6+15 points) Let us consider a sequence of $n \geq 2$ independent Bernoulli trials with probability $1/2$ ($n$ flips of a fair coin); the result of the experiment is a $(0, 1)$-string $Z$ of length $n$. Let $X$ denote the number of consecutive pairs of zeros in this string. (E.g., if $Z = 00100110001$ then $X = 4$.) (a) What is the size of the sample space for this experiment? (b) Determine $E(X)$. Prove. Give a clear definition of your variables. (c) Determine $\text{Var}(X)$. Your answers should be very simple closed-form expressions. Show all your work.

4. (4+5+3+6 points) (a) Define the relation $a_n = \Omega(b_n)$. Do not use the big-Oh notation. Give a properly quantified formula, no English words. (b) True or false? $F_{n+1} = O(F_n)$ (Fibonacci numbers). Give a simple proof of your answer. (c) True or false: $\ln x = \Theta(\log_2 x)$. Prove your answer. (d) True or false: $\pi(x) = \Omega(x^{0.9})$, where $\pi(x)$ is the number of primes $\leq x$.

5. (13 points) Let $n$ be a positive integer, $n \equiv -1 \pmod 6$. Pick a random integer $X$ from $\{1, 2, \ldots, n\}$. Let $A$ be the event that $X$ is even; and $B$ the event that $3 \mid X$. Are the events $A$ and $B$ positively correlated, negatively correlated, or independent? Prove your answer.
6. **(5+5 points)** Find each multiplicative inverse or prove that it does not exist: (a) $5^{-1} \pmod{7}$; (b) $6^{-1} \pmod{27}$.

7. **(15 points)** Prove: $\gcd(2^k - 1, 2^\ell - 1) = 2^d - 1$, where $d = \gcd(k, \ell)$. (Hint: induction on $k + \ell$.)

8. **(4+12 points)** (a) State Fermat’s little Theorem. (Do not prove.) (b) Let $f(x) = 1 + x + x^2 + \cdots + x^{29}$. Prove:

   $$\forall x \; (f(x) \equiv 0 \text{ or } \pm 1 \pmod{31})$$

9. **(6+6B points)** (a) Prove: a bipartite graph with $n$ vertices has at most $n^2/4$ edges. (b) **(BONUS)** Prove: a triangle-free graph on $n$ vertices has at most $n^2/4$ edges.

10. **(8+5 points)** Let $G$ be a regular graph of degree $r$ (every vertex has degree $r$) with $n$ vertices. Assume $G$ has girth at least 5 (there is no cycle of length shorter than 5). (a) Prove: $n \geq r^2 + 1$. (b) Draw and name a regular graph of degree $r = 3$, girth 5, and $n = r^2 + 1 = 10$ vertices.

11. **(1+1+5+5 points)** A graph is **self-complementary** if it is isomorphic to its complement. (a) Draw a self-complementary graph with 4 vertices. (b) Draw a self-complementary graph with 5 vertices. (c) Prove: if there is a self-complementary graph with $n$ vertices then $n \equiv 0 \text{ or } 1 \pmod{4}$. (d) Prove: if $G$ is self-complementary then $\chi(G) \geq \sqrt{n}$.

12. **(BONUS 4B points)** Let $G$ denote a (uniform) random graph on a given set of $n$ vertices. What is the probability that all vertices of $G$ have even degree? Your answer should be a simple closed-form expression.

13. **(BONUS 4B points)** Pick a random integer $x$ between 1 and $n$. Let $r(x)$ denote the number of distinct prime divisors of $x$ (so $r(12) = 2$). Prove: $E(r(x)) \sim \ln \ln n$. You may use the fact that $\sum_{p \leq x} 1/p \sim \ln \ln x$, where the summation is over primes.

14. **(BONUS 4B points)** Prove:

   $$\sum_{i=0}^{k} \binom{n}{i} \leq \left(\frac{ne}{k}\right)^k$$