CMSC-37110 Discrete Mathematics
MIDTERM EXAM October 29, 2010

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This exam contributes 16% to your course grade.

Do **not** *use book, notes.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else.

1. (8+8+8 points) Let X, Y, Z be random variables over a finite probability space. (a) Define what it means that X, Y, Z are independent. Do not include superfluous conditions.
   (b) Prove: if X, Y, Z are independent then they are pairwise independent.
   (c) Prove that the following statement is false: if X, Y, Z are pairwise independent then they are independent. Make your sample space small.

2. (2+9 points) We roll n (6-sided) dice. Let X denote the product of the n numbers shown. (a) What is the size of the sample space? (b) Determine $E(X)$. Your answer should be a very simple formula.

3. (1+6+15 points) Let us consider a sequence of $n \geq 2$ independent Bernoulli trials with probability $1/2$ (n flips of a fair coin); the result of the experiment is a $(0,1)$-string Z of length n. Let X denote the number of consecutive pairs of zeros in this string. (E.g., if $Z = 00100110001$ then $X = 4$.) (a) What is the size of the sample space for this experiment? (b) Determine $E(X)$. Prove. Give a clear definition of your variables. (c) Determine $\text{Var}(X)$. Your answers should be very simple closed-form expressions. Show all your work.

4. (4+5+3+6 points)
   (a) Define the relation $a_n = \Omega(b_n)$. Do not use the big-Oh notation. Give a properly quantified formula, no English words.
   (b) True or false? $F_{n+1} = O(F_n)$ (Fibonacci numbers). Give a simple proof of your answer.
   (c) True or false: $\ln x = \Theta(\log_2 x)$. Prove your answer.
   (d) True or false: $\pi(x) = \Omega(x^{0.9})$, where $\pi(x)$ is the number of primes $\leq x$.

5. (13 points) Let n be a positive integer, $n \equiv -1 \pmod 6$. Pick a random integer X from $\{1, 2, \ldots, n\}$. Let A be the event that X is even; and B the event that $3 \mid X$. Are the events A and B positively correlated, negatively correlated, or independent? Prove your answer.

6. (5+5 points) Find each multiplicative inverse or prove that it does not exist: (a) $5^{-1} \pmod 7$; (b) $6^{-1} \pmod{27}$.

7. (15 points) Prove: $\gcd(2^k - 1, 2^\ell - 1) = 2^d - 1$, where $d = \gcd(k, \ell)$. (Hint: induction on $k + \ell$.)

8. (4+12 points) (a) State Fermat's little Theorem. (Do not prove.) (b) Let $f(x) = 1 + x + x^2 + \cdots + x^{29}$. Prove:

$$(\forall x)(f(x) \equiv 0 \text{ or } \pm 1 \pmod{31}) \ .$$

9. (6+6B points) (a) Prove: a bipartite graph with n vertices has at most $n^2/4$ edges. (b) (BONUS) Prove: a triangle-free graph on n vertices has at most $n^2/4$ edges.

10. (8+5 points) Let G be a regular graph of degree r (every vertex has degree r) with n vertices. Assume G has girth at least 5 (there is no cycle of length shorter than 5). (a) Prove: $n \geq r^2 + 1$. (b) Draw and name a regular graph of degree $r = 3$, girth 5, and $n = r^2 + 1 = 10$ vertices.

11. (1+1+5+5 points) A graph is *self-complementary* if it is isomorphic to its complement. (a) Draw a self-complementary graph with 4 vertices. (b) Draw a self-complementary graph with 5 vertices. (c) Prove: if there is a self-complementary graph with n vertices then $n \equiv 0$ or $1 \pmod 4$. (d) Prove: if G is self-complementary then $\chi(G) \geq \sqrt{n}$.

12. (BONUS 4B points) Let $\mathcal{G}$ denote a (uniform) random graph on a given set of n vertices. What is the probability that all vertices of $\mathcal{G}$ have even degree? Your answer should be a simple closed-form expression.

13. (BONUS 4B points) Pick a random integer x between 1 and n. Let $r(x)$ denote the number of distinct prime divisors of x (so $r(12) = 2$). Prove: $E(r(x)) \sim \ln\ln n$. You may use the fact that $\sum_{p<x} 1/p \sim \ln\ln x$, where the summation is over primes.

14. (BONUS 4B points) Prove:

$$\sum_{i=0}^{k} \binom{n}{i} \leq \left(\frac{ne}{k}\right)^k \ .$$