CMSC-37110 Discrete Mathematics
SECOND QUIZ October 15, 2010

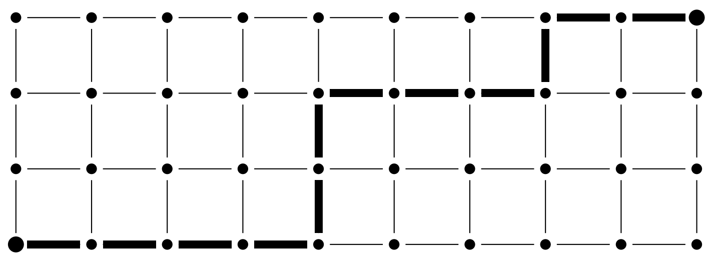
Name (print): ______________________________

Do **not** *use book, notes, scratch paper.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else. **Write your solution in the space provided.** You may CONTINUE ON THE REVERSE. This exam contributes 6% to your course grade.

1. (10 points) Find two sequences $\{a_n\}$ and $\{b_n\}$ of positive numbers such that $a_n \sim b_n$ but $a_n^n \neq \Theta(b_n^n)$. Prove your answer.

2. (8 points) Count the solutions of the equation $\sum_{i=1}^{k} x_i = n$ where the x_i are integers and $(\forall i)(x_i \geq 2)$. Your answer should be a simple closed-form expression (no summation symbols or dot-dot-dots). Prove. You may use a result proved in class.

3. (8 points) (Manhattan routing) Count the shortest paths between the bottom left corner and the top right corner of a $k \times \ell$ grid graph. (This graph has $k\ell$ vertices.) Notice that all shortest paths have length $k + \ell - 2$. Your answer should be a simple closed-form expression. Prove your answer by giving a bijection to objects counted in class. (The picture highlights one of these shortest paths in the 4×10 grid graph.)



4. (8+4B+4B points) Let $B(n)$ denote the number of equivalence relations on a set of n elements. (a) Prove: $B(n) \le n^n$. (b) (BONUS) Prove: $(\forall k \ge 0)(B(n) \ge k^{n-k})$. (c) (BONUS) Prove: $\ln B(n) \sim n \ln n$.

5. (6+6 points) Evaluate the following sums. Give closed-form expressions; do not prove. (a) $\sum_{k=0}^{n} 2^{-k/2}$ (b) $\sum_{k=0}^{n} \binom{n}{k} 2^{-k/2}$

6. (6 points) Prove: $k! \ge (k/\mathrm{e})^k$ for all $k \ge 0$.

7. (8 points) Use the preceding problem to prove: $\ln(n!) \sim n \ln n$. Do not use Stirling's formula.

8. (BONUS 5B points) Prove: the produt of k consecutive integers is always divisible by $k!$.