1. (12 points) Let $A$ be a stochastic matrix. Prove that every (complex) eigenvalue of $A$ has absolute value $\leq 1$.

2. (5 points) Draw a strongly connected digraph with period 3 but with no cycle of length 3. Use as few edges as possible. Don’t prove minimality.

3. (6 points) Give an example of a stochastic matrix of which the powers do not converge. Indicate why they don’t converge.

4. (12 points) Let $q = (q_1, \ldots, q_n)$ be a stationary distribution of an irreducible Markov Chain. Prove that all the $q_i$ are positive. (Recall that a Markov Chain is irreducible if the corresponding digraph is strongly connected.)
5. (8+7 points) (a) Let $A$ be a diagonalizable $n \times n$ matrix with characteristic polynomial $f_A(x) = (x - 1)^n$. Prove that $A$ is the identity matrix. (b) Give an example of a non-diagonalizable $2 \times 2$ matrix with characteristic polynomial $(x - 1)^2$.

6. (5+5 points) (a) Find a $2 \times 2$ real matrix which has no real eigenvalues. (b) Prove that every $3 \times 3$ real matrix has at least one real eigenvalue.

7. (BONUS, 8B points) Let $A$ be an $n \times n$ matrix. Suppose $v_1, \ldots, v_k$ are eigenvectors corresponding to distinct eigenvalues. Prove: $v_1, \ldots, v_k$ are linearly independent.

8. (BONUS, 8B points) Compute the value $d_n$ of the determinant of the following $n \times n$ matrix:

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & -1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1
\end{pmatrix}
$$

(Each diagonal entry and each entry just above the diagonal is 1, each entry just below the diagonal is $-1$; all other entries are 0.)