CMSC-37110 Discrete Mathematics
FOURTH QUIZ December 1, 2010

Name (print): ______________________________

Do **not** *use book, notes, scratch paper.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else. **Write your solution in the space provided.** You may CONTINUE ON THE REVERSE. This test contributes 6% to your course grade.

1. (12 points) Let A be a stochastic matrix. Prove that every (complex) eigenvalue of A has absolute value ≤ 1.

2. (5 points) Draw a strongly connected digraph with period 3 but with no cycle of length 3. Use as few edges as possible. Don't prove minimality.

3. (6 points) Give an example of a stochastic matrix of which the powers do not converge. Indicate why they don't converge.

4. (12 points) Let $q = (q_1, \ldots, q_n)$ be a stationary distribution of an irreducible Markov Chain. Prove that all the q_i are positive. (Recall that a Markov Chain is irreducible if the corresponding digraph is strongly connected.)

5. (8+7 points) (a) Let A be a diagonalizable $n \times n$ matrix with characteristic polynomial $f_A(x) = (x-1)^n$. Prove that A is the identity matrix. (b) Give an example of a non-diagonalizable 2×2 matrix with characteristic polynomial $(x-1)^2$.

6. (5+5 points) (a) Find a 2×2 real matrix which has no real eigenvalues. (b) Prove that every 3×3 real matrix has at least one real eigenvalue.

7. (BONUS, 8B points) Let A be an $n \times n$ matrix. Suppose $v_1, \ldots, v_k$ are eigenvectors corresponding to distinct eigenvalues. Prove: $v_1, \ldots, v_k$ are linearly independent.

8. (BONUS, 8B points) Compute the value d_n of the determinant of the following $n \times n$ matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$$

(Each diagonal entry and each entry just above the diagonal is 1, each entry just below the diagonal is -1; all other entries are 0.)