CMSC-37110 Discrete Mathematics
FINAL EXAM December 8, 2011

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This exam contributes 30% to your course grade.

Do **not** *use book, notes.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else.

1. (1+7 points) (a) Define the little-oh notation.
   (b) Prove: $n^{100} = o(1.01^n)$. Elegance counts. Do not use L'Hospital's rule beyond using the fact that $\lim_{x\to\infty} \ln x/x = 0$. (Hint: substitute a new variable.)

2. (12 points) Let J denote the $n \times n$ all-ones matrix (all entries are 1). What are the eigenvalues of this matrix, and their multiplicities? Describe an eigenbasis of this matrix.

3. (10 points) Prove: the matrix $N = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ does not have an eigenbasis.

4. (10 points) Find the complex eigenvalues and a complex eigenbasis of the rotation matrix $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

5. (10+6+18+14+10B points) (a) State the Spectral Theorem.
   (b) Define the (Euclidean) norm of an $n \times n$ real matrix B.
   (c) Prove: if A is a real symmetric $n \times n$ matrix then its norm is $\max |\lambda_i|$ where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A. Use the Spectral Theorem in the proof.
   (d) Prove: if B is an $n \times n$ real matrix (not necessarily symmetric) then the matrix $A = B^T B$ is symmetric and all its eigenvalues are nonnegative. (B^T is the transpose of B.)
   (e) (BONUS) Prove: if B is as in (d) then the norm of B is $\sqrt{\mu}$ where μ is the largest eigenvalue of $A = B^T B$.

6. (16 points) Let T be the transition matrix of a finite Markov chain. Prove: all eigenvalues of T have absolute value ≤ 1. (Note that these are complex numbers.)

7. (1+9 points) (a) Count the strings of length n over the alphabet $\{A, B, C, D, E\}$. (b) How many among these strings use all the five letters? Your answer should be a closed-form expression.

8. (20 points) We have a biased coin; the probability of "heads" is $1/3$. Consider the experiment that we flip the coin n times. We repeat this experiment n^2 times. Let $p(n)$ denote the probability that in each of the experiments, the number of heads is between $0.33n$ and $0.34n$. Prove: $1 - p(n)$ is exponentially small.

9. (8 + 6B points) (a) Construct three random variables that are pairwise but not fully independent. Make your sample space as small as possible.
(b) (BONUS) Construct n random variables that are $(n-1)$-wise but not fully independent. Make your sample space as small as possible.

10. (8+8 points) (a) Prove: the average degree of a planar graph is less than 6. (The average degree is the average of the degrees of all vertices.)
(b) Prove: for every positive ϵ there exists a planar graph with average degree $\geq 6 - \epsilon$.

11. (20 points) Prove: almost all graphs on n vertices have diameter 2.

12. (4 + 14 + 6 + 8 + 8 points) We have n guests and n gift items. For each gift item, we draw a guest's name at random. The same name can be drawn multiple times.
(a) What is the size of the sample space for this experiment?
(b) A guest is unlucky if his/her name is never drawn. Let X be the number of unlucky guests. Determine $E(X)$.
(c) Asymptotically evaluate your answer to (b). Give a very simple expression.
(d) Let p_n denote the probability that $X = 0$ (none of the guests is is unlucky). Detemine p_n (give a simple closed-form expression).
(e) True or false: $p_n < 1/2.7^n$ for all sufficiently large n. Prove your answer. (Note: $\mathrm{e} = 2.718\ldots$)

13. (8+8 points) (a) State the multinomial theorem (express $(x_1+\cdots+x_k)^n$ as a sum). Evaluate the coefficients.
(b) Count the terms in your expression. Your answer should be a very simple expression (a binomial coefficient).

14. (8 points) Let F_n denote the n-th Fibonacci number (starting with $F_0 = 0$, $F_1 = 1$). Prove: for all n, the numbers F_n and F_{n+2} are relatively prime.

15. (8+14 points) (a) Consider the infinite arithmetic progression $x_n = a + bn$ where a, b are positive integer constants. Prove: there exist two terms in the progression that are not relatively prime.
(b) Prove: there exists a 100-term arithmetic progression $y_n = c + dn$ $(n = 0, 1, \ldots, 99)$ where c, d are positive integer constants such that the 100 terms are pairwise relatively prime. Prove your answer. Do not use any results not proved in class.

16. (10+10 points) Let $a_n > 2$ and $b_n > 2$ be sequences of real numbers. Consider the following two statements:

. (1) $a_n = \Theta(b_n)$; (2) $\ln a_n \sim \ln b_n$.

(a) Prove that (2) does not follow from (1).

(b) Prove that if $a_n \to \infty$ then (2) follows from (1).

17. (16 points) Find an integer x between 1 and 30 such that for every integer $a \geq 0$ we have $a^x \equiv a^{7^{150}} \pmod{31}$. (The exponent is 7^{150}.) Do not use a calculator.

18. (BONUS 10B points) Let $n = pq$ where p, q are distinct primes. Prove that the following statement is false:

$(\forall a)(\text{if } \gcd(a, n) = 1 \text{ then } a^{n-1} \equiv 1 \pmod{n})$.

19. (BONUS 10B points) Let $A_1, \ldots, A_m$ be events such that $(\forall i)(P(A_i) = 1/2)$ and $(\forall i \neq j)(P(A_i \cap A_j) \leq 1/5)$. Prove: $m \leq 6$.