CMSC-37110 Discrete Mathematics
MIDTERM EXAM November 1, 2011

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This exam contributes 20% to your course grade.

Do **not** *use book, notes.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else.

1. (2+8+10 points) We roll n (6-sided) dice which have the numbers $0, 1, \ldots, 5$ written on their sides. Let X denote the product of the n numbers shown. (a) What is the size of the sample space? (b) Determine $E(X)$. Your answer should be a very simple formula. (c) Prove: the event "$X \geq E(X)$" is exponentially unlikely, i. e., $P(X \geq E(X)) \leq c^n$ for some constant $c < 1$ for all sufficiently large n. Clearly state your value c.

2. (18 points) Decide whether or not the following system of congruences is solvable. Prove your answer; do not solve. Show all your work. Do not use a calculator.

   $x \equiv 9 \pmod{56}$

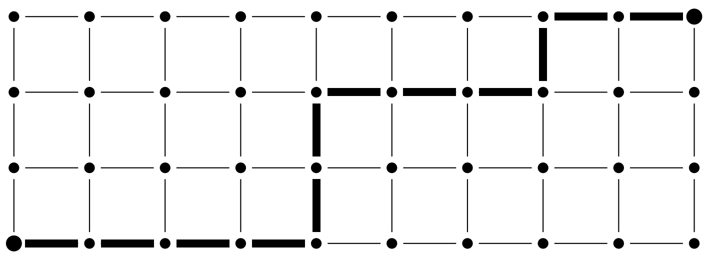
   $2x \equiv 4 \pmod{70}$

   $3x \equiv 11 \pmod{40}$

3. (16 points) Recall that $\pi(n)$ denotes the number of primes $\leq n$. Decide which of the following three statements, involving the little-oh notation, is true. Circle and prove your answer.
   (a) $\pi(n) = o(n^{0.9})$ (b) $n^{0.9} = o(\pi(n))$ (c) neither.

4. (3+6 points) (a) Count the graphs on a given set of n vertices. (b) How many among these will have exactly m edges? Your answers should be simple closed-form expressions involving binomial coefficients.

5. (12 points) Count the shortest paths from the bottom left corner to the top right corner of the $k \times \ell$ grid graph. (Example: The figure shows the 4×10 grid graph with a shortest path between the given corners highlighted.) Your answer should be a very simple closed-form expression (no summation or dot-dot-dots). Prove your answer.



6. (8+8 points) Let E_n denote the number of even subsets and O_n the number of odd subsets of a given set of n elements. Prove: for $n \geq 1$ we have $E_n = O_n$. Give (a) an algebra proof (use the Binomial Theorem); (b) a bijective proof (give a simple bijection between the set of even subsets and the set of odd subsets).

7. (5+15 points) Let us fix the positive integers k, ℓ, n where $k, \ell \leq n$. Let $A \subseteq [n]$ be a random subset of size k and $B \subseteq [n]$ a random subset of size ℓ (where $[n] = \{1, 2, \dots, n\}$). (a) What is the size of the sample space for this experiment? (b) Determine $E(|A \cap B|)$. (Hint: indicator variables.) Half the credit goes for a clear definition of your variables.

8. (12 points) A *Hamilton cycle* in a graph is a cycle that passes through each vertex. Prove that the 101×999 grid graph does not have a Hamilton cycle.

9. (4+5+4+6 points)

   (a) Define the relation $a_n = \Omega(b_n)$. Do not use the big-Oh notation. Give a properly quantified formula, no English words.

   (b) True or false? $F_{n+1} = O(F_n)$ (Fibonacci numbers). Give a simple proof of your answer.

   (c) True or false: $\ln x = \Theta(\log_2 x)$. Prove your answer.

   (d) True or false: $n! = O((n/\mathrm{e})^n)$. Prove your answer.

10. (16 points) Prove: all longest paths in a tree share a vertex.

11. (16 points) Construct two random variables X and Y over the same probability space such that $E(XY) = E(X)E(Y)$ but X and Y are not independent. Make your sample space small.

12. (7 points) Write 1 as a linear combination of 37 and 87. Show your work.

13. (12 points) Recall that the chromatic polynomial $f_G(x)$ of a graph G counts the legal colorings of G using the set $\{1, 2, \dots, x\}$ of colors (x is

a positive integer). Prove: if T is a tree with n vertices then $f_T(x) = x(x-1)^{n-1}$.

14. (7+6B points) (a) Prove: every bipartite graph with n vertices has at most $n^2/4$ edges. (b)(BONUS) Prove: every triangle-free graph with n vertices has at most $n^2/4$ edges. (A graph is *triangle-free* if it contains no K_3 subgraph.)

15. (BONUS 4B points) Let x be an even integer and p a prime divisor of x^2+1. Prove: $p \equiv 1 \pmod 4$. (Hint: Fermat's little Theorem.)

16. (BONUS 4B points) A graph is *self-complementary* if it is isomorphic to its complement. Prove: if G is self-complementary then $\chi(G) \geq \sqrt{n}$. (n is the number of vertices.)

17. (BONUS 5B points) Prove: the chromatic number of a triangle-free graph is $O(\sqrt{n})$ where n is the number of vertices.

18. (BONUS 4B points) Let $\mathcal{G}$ denote a (uniform) random graph on a given set of n vertices. What is the probability that all vertices of $\mathcal{G}$ have even degree? Your answer should be a simple closed-form expression.

19. (BONUS 4B points) Prove:

$$\sum_{i=0}^{k} \binom{n}{i} \leq \left(\frac{n\mathrm{e}}{k}\right)^k .$$