CMSC-37110 Discrete Mathematics
SECOND QUIZ October 20, 2011

Name (print): ____________________________________

Do **not** *use book, notes, scratch paper.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else. **Write your solution in the space provided.** You may CONTINUE ON THE REVERSE. This exam contributes 6% to your course grade.

1. (1+10 points) (a) Count the strings of length n over the alphabet $\{A, B, C, D, E\}$. (b) How many among these strings use all the five letters? Your answer should be a closed-form expression.

2. (3+6 points) (a) Consider a poker-hand, i. e., a set of 5 cards dealt from a well-shuffled standard deck of 52 cards. What is the probability that the five cards are of five different kinds? (There are 13 kinds of cards: aces, kings, queens, etc. There are 4 cards of each kind.) (b) Generalize the problem to a hand of t cards out of a deck of rk cards wich come in k kinds (r cards of each kind). Your answer should be a simple closed-form expression.

3. (6+4 points) For $n \geq 1$, prove:

$$\frac{4^n}{2n+1} < \binom{2n}{n} < 4^n.$$

Hint: what is the sum $\sum_{i=0}^{2n} \binom{2n}{i}$?

4. **(10 points)** Prove: $(\forall a)(a^{37} \equiv a \pmod{247}) \quad (247 = 13 \cdot 19)$

5. **(10 points)** What is the expected number of runs of k heads in a sequence of n coin flips? (A "run of k heads" means k consecutive heads. For instance, in the string $HHTHHHTTHTH$ there are 3 runs of 2 heads (H), starting at positions 1, 4, and 5, and 1 run of 3 heads.) Half the credit goes for the proper definition of your random variables. Your answer should be a simple closed expression.

6. **(5+5 points)** Evaluate the following sums. Give closed-form expressions; do not prove. (a) $\sum_{k=0}^{n} 2^{-k/2}$ (b) $\sum_{k=0}^{n} \binom{n}{k} 2^{-k/2}$

7. **(BONUS PROBLEM: 8B points)** Let p be an odd prime. Suppose for some x we have $x^2 \equiv -1 \pmod{p}$. Prove: $p \equiv 1 \pmod{4}$.

8. **(BONUS PROBLEM: 6B points)** Prove: the produt of k consecutive integers is always divisible by $k!$.