CMSC-37110 Discrete Mathematics
THIRD QUIZ November 29, 2011

Name (print): ____________________________________

Do **not** *use book, notes, scratch paper.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else. **Write your solution in the space provided.** You may CONTINUE ON THE REVERSE. This exam contributes 6% to your course grade.

1. (12 points) We have a biased coin; the probability of "heads" is $1/3$. Consider the experiment that we flip the coin n times. We repeat this experiment n^2 times. Let $p(n)$ denote the probability that in each of the experiments, the number of heads is between $0.33n$ and $0.34n$. Prove: $1 - p(n)$ is exponentially small.

2. (5 points) True or false: if A, B are 2×2 real matrices then $\det(A+B) = \det(A) + \det(B)$. State and prove your answer.

3. (6+5 points) Let G be a 4-regular bipartite graph. ("4-regular" means every vertex has degree 4.) (a) Prove: G is not planar. (b) Draw a counterexample if we drop the condition that G is bipartite. Use as few vertices as possible.

4. (5 points) Draw a strongly connected aperiodic digraph with no cycles of length ≤ 3. Use as few edges as possible.

5. (6 points) Prove: if a finite Markov chain has two stationary distributions then it has infinitely many.

6. (9 points) Determine the rank of the $n \times n$ matrix $B = (b_{i,j})$ where $b_{i,j} = i + j$. Prove your answer.

7. (12 points) Prove: if a digraph G is not strongly connected then it has a cut. (A cut is a partition $V = A \dot{\cup} B$ of the set of vertices such that A, B are nonempty and there is no edge from B to A.)

8. (BONUS PROBLEM: 4B points) Prove: if A, B are $n \times n$ real matrices then $AB - BA \neq I$ (where I denotes the identity matrix).