1. (12 points) We have a biased coin; the probability of “heads” is 1/3. Consider the experiment that we flip the coin \( n \) times. We repeat this experiment \( n^2 \) times. Let \( p(n) \) denote the probability that in each of the experiments, the number of heads is between \( 0.33n \) and \( 0.34n \). Prove: \( 1 - p(n) \) is exponentially small.

2. (5 points) True or false: if \( A, B \) are \( 2 \times 2 \) real matrices then \( \det(A + B) = \det(A) + \det(B) \). State and prove your answer.

3. (6 + 5 points) Let \( G \) be a 4-regular bipartite graph. (“4-regular” means every vertex has degree 4.) (a) Prove: \( G \) is not planar. (b) Draw a counterexample if we drop the condition that \( G \) is bipartite. Use as few vertices as possible.
4. **(5 points)** Draw a strongly connected aperiodic digraph with no cycles of length $\leq 3$. Use as few edges as possible.

5. **(6 points)** Prove: if a finite Markov chain has two stationary distributions then it has infinitely many.

6. **(9 points)** Determine the rank of the $n \times n$ matrix $B = (b_{i,j})$ where $b_{i,j} = i + j$. Prove your answer.

7. **(12 points)** Prove: if a digraph $G$ is not strongly connected then it has a cut. (A cut is a partition $V = A \cup B$ of the set of vertices such that $A, B$ are nonempty and there is no edge from $B$ to $A$.)

8. **(BONUS PROBLEM: 4B points)** Prove: if $A, B$ are $n \times n$ real matrices then $AB - BA \neq I$ (where $I$ denotes the identity matrix).