CMSC-27410/37200 Honors Combinatorics
FINAL EXAM March 12, 2012

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This exam contributes 38% to your course grade. Recall that the midterm contributes 20%, homework 38%, and class participation 4%. Take this problem sheet home as a souvenir.

Do **not** *use book, notes.* **Show all your work.** If you are not sure of the meaning of a problem, or you are not sure whether or not you can use a result without proof, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else.

1. (12+8+15 points)

   (a) Define the Shannon capacity $\Theta(G)$ of a graph G as the limit of certain sequence associated with G. Define the graph-product concept involved in the definition.

   (b) Recall *Fekete's Lemma:* If the sequence $\{a_k\}$ of positive numbers satisfies $a_{r+s} \geq a_r a_s$ for all $r, s \geq 1$ then $\lim_{n\to\infty} a_n^{1/n}$ exists. - Use Fekete's Lemma to show that the limit in part (a) exists.

   (c) Prove: if G is self-complementary (isomorphic to its complement) then $\Theta(G) \geq \sqrt{n}$, where n is the number of vertices of G.

2. (5+5+8+12 points)

   (a) Define the quantity ν^* (fractional matching number) for a hypergraph.

   (b) Let $\mathcal{H} = (P, L, I)$ be a projective plane of order k. (Recall: every line has $k+1$ points.) View $\mathcal{H}$ as a hypergraph. (The points are the vertices, the lines correspond to the edges.) Calculate (b1) $\nu(\mathcal{H})$, (b2) $\nu^*(\mathcal{H})$. (b3) Prove: $\tau(\mathcal{H}) = k+1$.

3. (20+15 points) In a (± 1)-matrix, every entry is 1 or -1. Recall: an *Hadamard matrix* is an $n \times n$ (± 1)-matrix of which the rows are orthogonal with respect to the standard dot product.

   (a) Prove Lindsay's inequality: if T is a $k \times \ell$ submatrix of an $n \times n$ Hadamard matrix and S is the sum of the entries of T then $|S| \leq \sqrt{k\ell n}$.

   (b) Prove that in the Gale-Berlekamp switching game on an $n \times n$ board, Player I can ensure that the payoff will not be greater than $O(n^{3/2})$.

   In this game, Player I chooses an $n \times n$ (± 1)-matrix A and Player II selects a set of rows and a set of columns and switches the signs of each selected row and then switches the signs of each selected column. The payoff is the sum of all entries of the resulting matrix B. You may use without proof the fact that for every $k \geq 1$ there exists an Hadamard matrix of size $2^k \times 2^k$. However, n is not necessarily a power of 2 (but you get partial credit if you solve the case when it is).

4. (12 points) Determine the fractional covering number τ^* of P_n, the path of length $n-1$. (P_n has n vertices and $n-1$ edges. Make sure you don't confuse n and $n-1$.)

5. (15+10 points)

   (a) Let τ be the covering number and χ the chromatic number of the graph G. Prove: $\chi \leq \tau + 1$.

   (b) Prove: for every positive integer k there exists a graph such that $\tau = k$ and $\chi = k + 1$.

6. (15+18 points)

   (a) Prove the Kővári - Sós - Turán Theorem: If a graph G has no 4-cycle then $m = O(n^{3/2})$. (As usual, n denotes the number of vertices and m the number of edges.)

   (b) Prove: If a graph has no 4-cycle then its chromatic number is $O(\sqrt{n})$.

7. (15 points) *Cayley's formula* says that the number of trees on a given set of n vertices is n^{n-2}. Use Cayley's formula to prove that for all sufficiently large n, there are more than 2.7^n non-isomorphic trees with n vertices.

8. (90 points) (Erdős) Prove: there exists a positive constant α such that for all sufficiently large n there exists a triangle-free graph G with n vertices and chromatic number $> n^\alpha$. Give your best lower bound for α.

9. (15+8+7 points) (Quadratic character) Let p be an odd prime number. Recall the following definitions. We say that $a \equiv b \pmod{m}$ if $m \mid a-b$ (m divides $a - b$). An integer a is a *quadratic residue* mod p if $a \not\equiv 0 \pmod{p}$ and there exists an integer x such that $a \equiv x^2 \pmod{p}$; and a is a *quadratic nonresidue* mod p if $a \not\equiv x^2 \pmod{p}$ for any x. The mod-p quadratic character χ_2 is defined on integers a by setting $\chi_2(a) = 0$ if $a \equiv 0 \pmod{p}$; and if $a \not\equiv 0 \pmod{p}$ then $\chi_2(a) = 1$ if a is a quadratic residue mod p and $\chi_2(a) = -1$ if a is a quadratic non-residue mod p.

   (a) Prove that χ_2 is multiplicative: for any pair (a, b) of integers, $\chi_2(ab) = \chi_2(a)\chi_2(b)$. Use only basic facts about prime numbers;

do not use the existence of a primitive root mod p. Hint: count the quadratic residues among the numbers $\{1, 2, \ldots, p-1\}$.

(b) Use Fermat's little Theorem to prove that if $p \equiv -1 \pmod 4$ then -1 is a quadratic nonresidue mod p. (Fermat's little Theorem asserts that for any prime p and any integer $a \not\equiv 0 \pmod p$ we have $a^{p-1} \equiv 1 \pmod p$.)

(c) Recall that for a prime $p \equiv -1 \pmod 4$, the *Paley tournament* $P(p)$ is defined as follows: the vertices of $P(p)$ are the integers $\{0, 1, \ldots, p-1\}$; we draw an $i \to j$ arrow if $i - j$ is a quadratic residue mod p. Prove: this rule indeed defines a tournament.

10. (75 points) (R. L. Graham and J. H. Spencer: explicit k-paradoxical tournaments) Prove that there exists a constant c such that for all primes $p \equiv -1 \pmod 4$ satisfying $p > ck^2 4^k$, the Paley tournament $P(p)$ is k-paradoxical, i. e., every set of k players is beaten by some player.

    Use Weil's character-sum estimate. Let f be a polynomial of degree $d \geq 1$ with integer coefficients whose lead coefficient is not divisible by p and assume $f \not\equiv ag^2 \pmod p$ for any constant a and any polynomial g with integer coefficients. Then Weil's character-sum estimate asserts, for the case of the quadratic character mod p, that

$$\left| \sum_{x=0}^{p-1} \chi_2(f(x)) \right| \leq (d-1)\sqrt{p}.$$

11. (BONUS: 4+10+3+3 bonus points) Let G be a graph with n vertices, m edges, and t triangles. Let λ denote the largest eigenvalue of the adjacency matrix of G.

    (a) Prove: $\lambda \leq \sqrt{2m}$.

    (b) Prove: $t = O(m^{3/2})$. (Do not use (c).)

    (c) Prove: $t \leq cm^{3/2}$ where $c = \sqrt{2}/3$. (If you solve this, you also get the credit for (b).)

    (d) Prove: the constant $c = \sqrt{2}/3$ is best possible.

12. (BONUS: 20 bonus points) Prove: If the graph G contains no 5-cycle then its chromatic number is $O(\sqrt{n})$ (where n is the number of vertices).

Total 380 points + 40 bonus points