CMSC-27410/37200 Honors Combinatorics
MIDTERM EXAM February 24, 2012

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This exam contributes 20% to your course grade.

Do **not** *use book, notes.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else.

1. (22 points) Prove: the chromatic number of a triangle-free graph is $O(\sqrt{n})$ where n is the number of vertices.

2. (18 points) Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denote the eigenvalues of the adjacency matrix of the graph G. Prove: $\sum_{i=1}^{n} \lambda_i^2 = 2m$ where m denotes the number of edges.

3. (21 points) Prove: for almost all graphs G, we have $\chi(G) > (\omega(G))^{100}$, where $\omega(G)$ denotes the clique number (size of largest complete subgraph) and $\chi(G)$ denotes the chromatic number of G.

4. (16 points) Let $r = 2s + t$ where r, s, t are positive integers. The Kneser graph $K(r, s)$ has $\binom{r}{s}$ vertices corresponding to the s-element subsets of a set of r elements; adjacency corresponds to disjointness. Recall that the odd-girth of a graph is the length of its shortest off cycle. Prove that the odd-girth of $K(r, s)$ is $\geq r/t$.

5. (4+15 points) (a) Count the graphs on a given set of n vertices. (b) Show that there are at least $2^{\binom{n}{2}}/n!$ nonisomorphic graphs on n vertices.

6. (5+18 points) (a) Define the quantity τ^* (fractional covering number) for a hypergraph. (b) Let $H = (V, E)$ be a regular k-uniform hypergraph. (k-uniformity means every edge has k vertices; regularity means every vertex belongs to the same number of edges.) Prove: $\tau^* = n/k$.

7. (5+16 points) Let us fix the positive integers k, ℓ, n where $k, \ell \leq n$. Let $A \subseteq [n]$ be a random subset of size k and $B \subseteq [n]$ a random subset of size ℓ (where $[n] = \{1, 2, \ldots, n\}$). (a) What is the size of the sample space for this experiment? (b) Determine $E(|A \cap B|)$. (Hint: indicator variables.) Half the credit goes for a clear definition of your variables.

8. (20 + 3 points) Recall that a tournament is an oriented complete graph. (Every edge is oriented in exactly one direction.) The vertices represent "players" and an edge $u \to v$ indicates that player u beat player v. We say that a tournament is k-paradoxical if to every set of k players there is a player who beat all those k players. (a) Prove that for every k the following hold: almost all tournaments are k-paradoxical. ("Almost

all" refers to the probability approaching 1 as $n \to \infty$. Our model is the uniform distribution over orientations of the complete graph: we flip a coin for each edge of the complete graph to decide its orientation.) (b) State the size of the sample space when the tournament has n vertices.

9. (20 + 5 points) (a) For a graph G and a positive integer x let $f_G(x)$ denote the number of legal colorings of G from the set $\{1, \dots, x\}$ of colors. (Not all colors need to be used.) Prove: for every graph, $f_G(x)$ is a polyinomial in x (the "chromatic polynomial"). (b) Compute the chromatic polynomial of the path of length n.

10. (12 points) A graph is *self-complementary* if it is isomorphic to its complement. Prove: if G is self-complementary then $\chi(G) \geq \sqrt{n}$. (n is the number of vertices.)

11. (BONUS 8B points) Let $S(n, 5) = \sum_{k=0}^{\infty} \binom{n}{5k}$. Prove: there exists a constant $c > 0$ such that for all sufficiently large n we have

$$\left| S(n, 5) - \frac{2^n}{5} \right| < (2 - c)^n.$$

12. (BONUS 5B points) Let $\mathcal{G}$ denote a (uniform) random graph on a given set of n vertices. What is the probability that all vertices of $\mathcal{G}$ have even degree? Your answer should be a simple closed-form expression.

13. (BONUS 5B points) Prove:

$$\sum_{i=0}^{k} \binom{n}{i} \leq \left(\frac{ne}{k} \right)^k .$$

Total 200 points + 18B bonus