

1. **(Heads vs tails)** We flip a fair coin  $n$  times; let  $X$  denote the number of heads and  $Y$  the number of tails. Prove that these two quantities are likely to be  $\Omega(\sqrt{n})$  apart. More precisely, prove that  $\Pr(|X - Y| \leq k) \leq (2k + 1)/\sqrt{n}$  ( $k \geq 0$ ).  
 Note. This is not an asymptotic result, it is true for all  $n \geq 1$  and  $k \geq 0$ . Use the inequality  $\binom{n}{t} \leq 2^n/\sqrt{n}$  which is true for all  $n \geq 1$  and  $t \geq 0$  (see MN, pp 84-85).
2. **(Picking coins: Who has the advantage?)** A row of 101 coins is placed on a table. Alice and Bob take turns pocketing a coin from one of the ends (they choose each time, which end). (a) Prove: Alice *cannot* necessarily ensure that she gets at least as much money as Bob (even though she gets to pick one more coin than Bob). (b) Prove the same under the restriction that the value of each coin is 1 or 2 units. (c) Suppose there are  $2n + 1$  coins, and the game master decides by flipping a fair coin  $2n + 1$  times which position gets a 1 and which gets a 2. Prove: for large  $n$ , it is very likely that Bob gets to pocket more than Alice. (Recall that with an even number of coins, Alice always has a way of getting at least as much as Bob.)
3. **(Cleaning the corner)** We label the cells of the positive quadrant (the “game board”) by pairs of integers  $\{(i, j) : i, j \geq 0\}$ . The neighbor to the North of cell  $(i, j)$  is cell  $(i + 1, j)$ ; the neighbor to the East is cell  $(i, j + 1)$ . The corner cell is  $(0, 0)$ . The Manhattan distance between cells  $(i_1, j_1)$  and  $(i_2, j_2)$  is  $|i_1 - i_2| + |j_1 - j_2|$ .  
 Chips are placed on some of the cells, at most one chip per cell. Chips “shift and multiply” in the following manner: suppose a chip is on cell  $(i, j)$ . If both its neighbor to the North and its neighbor to the East are empty, we can remove the chip from  $(i, j)$  and place a chip on its neighbor to the North and another chip on the neighbor to the East.  
 Initially we put a chip on cell  $(0, 0)$ ; otherwise the game board is empty. We wish to clean the corner, i. e., we wish to achieve, by a sequence of “shift/multiply” moves, that there be no chip left within Manhattan distance  $d$  from the corner. Prove that this is impossible (a) for  $d = 3$ ; (b) for  $d = 2$ .
4. **(Spreading Infection)** Some of the 64 cells of a chessboard are initially infected. Subsequently the infection spreads according to the following rule: if two neighbors of a cell are infected then the cell gets infected. No cell is ever cured. What is the minimum number of cells that need to be initially infected to guarantee that the infection spreads all over the chessboard? It is easy to see that 8 are sufficient in many ways. Prove that 7 are not enough. This is an AH-HA problem.