

Math Puzzles - Third Set. April 10, 2007

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1. **(Eventown)** (a) Prove: every maximal (not extendable) set of Eventown clubs is maximum ($= 2^{\lfloor n/2 \rfloor}$). (b) Prove: there exist maximum Eventown club systems that are not isomorphic to a “married couples” system.
2. **(Monotone subsequences: Erdős – Szekeres)** Prove: a sequence of $k\ell + 1$ distinct real numbers necessarily contains either an increasing sequence of $k + 1$ or a decreasing sequence of $\ell + 1$ terms. (AH-HA)
3. **(Hilbert matrix)** Let $a_1, \dots, a_n, b_1, \dots, b_n$ be $2n$ distinct elements of a field F . Prove that the $n \times n$ matrix $H = (h_{ij})$ is nonsingular, where

$$h_{ij} = \frac{1}{a_i - b_j}.$$

4. **(k -wise independent random variables)** A set of m random variables is k -wise independent if every k of them are independent.
(a) Construct a probability space and n 3-wise independent coin flips ($(0, 1)$ -variables with each outcome having probability $1/2$) over a sample space of size $O(n)$. (Hint: assume $n = 2^k$; make $|\Omega| = 2n$.)
(b) Prove: If m 4-wise independent random variables exist then the sample space has size $\Omega(m^2)$. (Hint: prove that the sample space has size $\geq \binom{m+1}{2}$. Use the linear algebra method.)
5. **(Guess your hat)** An athletic team has n members; their shirts are numbered 1 to n . Each member receives a hat; each hat is either red or blue. The hats are distributed at random, with a coin flip for each person. Nobody knows the color of their own hat but they can see everybody else’s hat and they also see everybody’s number (printed on the shirt) (including their own). Team members must guess the color of their hat or pass under the following rules. Each team member must simultaneously and independently check “red” or “blue” or “pass” on an answer sheet (no communication permitted). If everybody passes, the team loses. If at least one person guesses the wrong color, the team loses. If at least one person does not pass and each person who does not pass guesses the right color, the team wins a big prize. Before the hats are distributed, the team gets to have a long strategy session; their goal is to devise a strategy which maximizes the team’s chance of winning.

It should be clear that each person’s guess is wrong 50% of the time, regardless of strategy. So what’s the use of strategy? Can the team beat the odds?

Surprisingly, yes. In fact, the winning chance approaches 1 as the team grows. Prove: for $n = 2^k - 1$, it is possible to achieve a winning chance of $1 - 1/(n + 1)$. (Prove this for $n = 3$ first; the probability of winning should then be $3/4$.) (Hint: pairwise independence, perp.)

6. **(Erdős)** Let us consider a set of n points in the plane; assume the distance between each pair is at most 1 unit. Prove: the number of pairs at unit distance is $\leq n$.
7. **(Sylvester)** Let us consider n lines in the plane, not all of which pass through a point. Prove: there is a point in which exactly two of the lines intersect. (AH-HA solution found by Gallai about 70 years after the problem was posed.)
8. **(Diagonals)** Prove: the diagonals of a convex n -gon intersect in at most $\binom{n}{4}$ points. (AH-HA)
9. **(Eszter Klein)** Consider 5 points in the plane, no 3 of which are on a line. Prove: 4 of them form a convex quadrilateral.
10. **(Erdős – Szekeres: Happy Ending Problem)** Prove that for every n there is an N such that among any N points in the plane, no three of which are on a line, one can find n which form a convex n -gon. (Hint: Ramsey)