SOLUTIONS TO EXERCISE SET #4

SCOTT MESSICK*

Exercise 4.1.

(a) Define legal colorings of a hypergraph.
(b) Prove: if an r-uniform hypergraph H has $m \leq 2^{r-1}$ edges then H is 2-colorable.

Solution.

(a) Let $H = (V, E)$ be a hypergraph. A coloring is a function $f : V \to C$ where C is the set of "colors" (any set); f is a *legal coloring* if for each edge $e \in E$, $|f(e)| > 1$, i.e., there exist vertices $u, v \in E$ such that $f(u) \neq f(v)$. In other words, no edge is monochromatic.
(b) First of all, the statement is false for $r = 1$: an edge of size 1 has no legal coloring. So we are going to assume $r \geq 2$.

Pick a random (not necessarily legal) 2-coloring of H. The probability space consists of all possible functions $f : V \to \{0, 1\}$, uniformly weighted. (So the size of the sample space is 2^n where $n = |V|$.) For each edge e, let B_e be the event that e is monochromatic. Let $A = \bigcup_e B_e$, the event that the coloring is illegal. We need to show that $Pr(A) < 1$.

Now $Pr(B_e) = 2/2^r = 1/2^{r-1}$. Therefore, by the union bound,

$$Pr(A) \leq \sum_{e \in E} Pr(B_e) = m/2^{r-1} \leq 1. \tag{1}$$

In fact, if $m < 2^{r-1}$ then the rightmost inequality is strict and we are done.

But the assumption was $m \leq 2^{r-1}$. In this case the only way we could have $Pr(A) = 1$ if we had equality everywhere in (1). This would require equality in the union bound, which can only happen if the events in question do not overlap. In our case, assuming $m \geq 2$, the events B_e are not pairwise disjoint because, for example, the elementary event that all vertices are assigned color 1 belongs to all the B_e.

The remaining case: $1 = m = 2^{r-1}$, so $r = 1$, which we did not allow. □

Exercise 4.2. Recall from class that an $n \times n$ matrix A is *fully indecomposable* if it does not contain a $k \times (n-k)$ all-zero submatrix for any k $(1 \leq k \leq n-1)$. Prove: if A is a nonnegative, fully indecomposable matrix then so is $A^T A$.

Solution. In fact we prove the more general statement that

Date: April 24, 2010.

*Slightly revised by instructor.

(∗) if A and B are nonnegative, fully indecomposable $n \times n$ matrices then so is AB.

We prove the contrapositive: if AB is not fully indecomposable, we prove that either A or B must also not be fully indecomposable. So let us assume that for some k, the matrix AB has a $k \times (n-k)$ all-zero submatrix. The entries of AB are $(a_i \cdot b_j)$ where $a_1, \ldots, a_n \in \mathbb{R}^n$ are the rows of A and $b_1, \ldots, b_n \in \mathbb{R}^n$ are the columns of B. Thus we have a collection of k rows of A and collection of $n-k$ columns of B such that any respective dot product is zero. Now, $a_i \cdot b_j = 0$ if and only if the sets of indices of nonzero entries of a_i and b_j, respectively, are disjoint, since all entries are nonnegative. Let $J \subseteq [n]$ denote the set of those column indices where any of our k rows a_i have a nonzero entry; and let $I \subseteq [n]$ denote the set of those row indices where any of our $n-k$ columns b_j have a nonzero entry. Our conclusion is that $I \cap J = \emptyset$.

There are two cases: either $|J| \leq k$ or $|I| \leq n-k$ (or both). In the first case we found a $k \times (n-k)$ all-zero submatrix in A; in the second case, a $k \times (n-k)$ all-zero submatrix in B. □

Exercise 4.4. Let A be a nonnegative, irreducible $n \times n$ matrix. Let x be a nonnegative eigenvector of A. Prove: x is positive.

Solution. Recall that an $n \times n$ matrix A is *irreducible* if it does not contain a $k \times (n-k)$ all-zero submatrix that does not intersect the diagonal (the set of row indices and column indices are disjoint). By Exercise 4.3, this is equivalent to the statement that the digraph associated with A is strongly connected. (The vertex set of this digraph is $[n]$ and by definition we have $i \to j$ exactly if $a_{ij} \neq 0$.)

Breaking the equation $Ax = \lambda x$ down componentwise, we have that for each i,

$$\lambda x_i = \sum_{j=1}^{n} a_{ij} x_j.$$

Since A and x are nonnegative, *any* positive value for a term on the right-hand side implies $x_i > 0$. So whenever $i \to j$ in the digraph, we have the implication $x_j > 0 \implies x_i > 0$. By induction it follows that if $x_j > 0$ and there is a directed path from i to j then $x_i > 0$. But since the digraph is strongly connected and $x_j > 0$ for at least one j, we have $x_i > 0$ for all i, as desired. □

Exercise 4.5. Let A be a nonnegative, irreducible $n \times n$ matrix. Let x, y be nonnegative eigenvectors. Prove: x is a scalar multiple of y.

Solution. Let λ and μ be the associated eigenvalues, so $Ax = \lambda x$ and $Ay = \mu y$. Assume without loss of generality $\lambda \leq \mu$. Since by the previous problem both x and y are strictly positive, we can assume by replacing y with a scalar multiple if necessary that $x \geq y$ and furthermore $x_i = y_i$ for some i. For that i, we then have

$$\lambda x_i = \sum_{j=1}^{n} a_{ij} x_j \geq \sum_{j=1}^{n} a_{ij} y_j = \mu y_i \geq \lambda y_i = \lambda x_i.$$

All these terms are therefore equal; in particular, $\lambda = \mu$. Consequently, $A(x-y) = \lambda(x-y)$ but since $x-y$ is nonnegative with a zero entry, by the previous problem, it cannot be an eigenvector so in fact $x - y = 0$ and we are done. □