CMSC-37110 Discrete Mathematics
FINAL EXAM December 11, 2012

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This exam contributes 30% to your course grade.

Do **not** *use book, notes.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else.

1. (1+1+2+5+4+10 points) Recall that a symmetric real matrix $A \in M_n(\mathbb{R})$ is *positive semidefinite* if $(\forall x \in \mathbb{R}^n)(x^T Ax \geq 0)$.

   (a) Define positive definite matrices.

   (b) Define indefinite matrices.

   (c) Give an example of an indefinite 2×2 matrix.

   (d) Let J denote the $n \times n$ all-ones matrix (every entry is 1). Prove that J is positive semidefinite.

   (e) Which diagonal matrices are positive definite? State, prove.

   (f) Let $A = (\alpha_{ij}) \in M_n(\mathbb{R})$ be an $n \times n$ matrix such that $(\forall i)(\alpha_{ii} > 1)$ and $(\forall i \neq j)(\alpha_{ij} = 1)$. (The diagonal entries are greater than 1 and the off-diagonal entries are equal to 1.) Prove that A is positive definite.

2. (14 points) True or false (state clearly). If **"false," give a simple counterexample.** If "true," do not prove. (Lose 1 point per wrong or missing answer and 2 points per wrong or missing counterexample. Lose 1 point for an unnecessarily complicated counterexample.)

   (a) If $\varphi : V \to W$ is a linear map and $v_1, \ldots, v_k$ are linearly independent vectors in V then $\varphi(v_1), \ldots, \varphi(v_k)$ are linearly independent vectors in W. **T F**

   (b) If $\varphi : V \to W$ is a linear map and $v_1, \ldots, v_k$ are linearly dependent vectors in V then $\varphi(v_1), \ldots, \varphi(v_k)$ are linearly dependent vectors in W. **T F**

   (c) If A, B are matrices and both AB and BA are defined then A and B are square matrices. **T F**

   (d) If A, B are positive matrices (all entries are positive) then $\text{rk}(A + B) \geq \text{rk}(A)$. **T F**

   (e) If we multiply a square matrix by the scalar β then the value of its determinant is multiplied by β. **T F**

   (f) The uniform distribution is a stationary distribution of every finite Markov Chain. **T F**

3. (2+8 points) Recall that the operator norm of a real matrix $A \in \mathbb{R}^{k\times\ell}$ is defined as $\max ||Ax||/||x||$ where the maximum is taken over all vectors $x \in \mathbb{R}^\ell$, $x \neq 0$.

   (a) Let α_{ij} be an entry of the matrix $A \in \mathbb{R}^{k\times\ell}$. Prove: $||A|| \geq |\alpha_{ij}|$.

   (b) Let B be a symmetric real matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Prove: $||B|| = \max_{1\leq i\leq n} |\lambda_i|$.

4. (2+1+9 points) A matrix is called *orthogonal* if its columns form an orthonormal basis of $\mathbb{R}^n$.

   (a) Describe the matrix of the rotation of the plane by θ with respect to the usual basis (a pair of orthogonal unit vectors).

   (b) Prove that this matrix is orthogonal.

   (c) Prove: if an $n \times n$ matrix A is orthogonal then its operator norm is 1.

5. (5+3 points) Let $A \in M_n(\mathbb{R})$. Assume $A^T = -A$. (a) Prove: if n is odd then $\det(A) = 0$. (b) Prove that this is not true for $n = 2$.

6. (8 points) Prove: for all sufficiently large n, the number of non-isomorphic trees with n vertices is greater than 2.7^n. Use Cayley's formula n^{n-2}; state what it is that this formula counts.

7. (8 points) Let A be the adjacency matrix of a graph. (So A is a symmetric real matrix.) Let λ_1 be the largest eigenvalue of A. Prove: $\lambda_1 \geq 2m/n$ where m is the number of edges, so $2m/n$ is the average degree.

8. (12 points) Let T be the transition matrix of an irreducible Markov Chain (i. e., the digraph of possible transitions is strongly connected). Let $\pi = (\pi_1, \ldots, \pi_n)$ be a stationary distribution for the Markov Chain. Prove: $(\forall i)(\pi_i > 0)$. Do not use the Perron - Frobenius Theorem.

9. (12 points) Prove: if λ is an eigenvalue of a stochastic matrix then $|\lambda| \leq 1$. (Note that λ is a complex number.) Do not use the Perron - Frobenius Theorem.

10. (12 points) Let p be an odd prime and $k \not\equiv 0 \pmod p$. Prove:

$$k^{(p-1)/2} \equiv \pm 1 \pmod p.$$

11. (15 points) In a well-shuffled deck of n cards, numbered 1 through n, what is the probability that cards #1 and #2 come next to each other (in either order)? Your answer should be an extremely simple expression; make it as simple as possible. Prove your answer.

12. (2+6 points) Find (a) the characteristic polynomial and (b) the minimal polynomial of the matrix $N = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Prove your answer to (b).

13. (5 points) For what values of the integer m is the following statement true:

$$(\forall x, y)(xy \equiv 0 \pmod{m} \Rightarrow (x \equiv 0 \pmod{m} \text{ or } y \equiv 0 \pmod{m}))$$

Prove your answer.

14. (3+5 points)

    (a) Define the Erdős - Rado arrow symbol $n \to (k, \ell)$.

    (b) Prove: $(\forall r, s \geq 1)(rs \not\to (r+1, s+1))$.

15. (6 points) Construct a probability space and a nonnegative random variable X such that $E(X) = 100$ but $P(X > 1) \leq 1/33$. Make your sample space as small as possible.

16. (6+8+5+6+6B points)

    (a) Draw the diagram of a weakly connected finite Markov Chain which has more than one stationary distribution. Give two stationary distributions for your Markov Chain. Use as few states as possible.

    (b) Recall that a finite Markov Chain is *irreducible* if its transition digraph is strongly connected. Draw the diagram of a reducible (not irreducible) finite Markov Chain with a unique stationary distribution. State the stationary distribution. Do not prove. Use as few states as possible.

    (c) Define ergodicity of a finite Markov Chain. Define the terms used in the definition in terms of directed graph concepts.

    (d) Draw the diagram of an irreducible but non-ergodic finite Markov Chain.

    (e) (BONUS) Prove: the stationary distribution of an irreducible finite Markov Chain is unique. (Prove uniqueness only. Do not prove the existence of a stationary distribution.) Do not use the Frobenius-Perron Theorem.

17. (8 points) Consider a deck of $4n$ cards, containing n kinds of cards, 4 of each kind. We pick a hand of k cards. What is the probability that no two of the cards are of the same kind? Give a simple closed-form expression using binomial coefficients.

18. (10+8 points)

(a) Consider the simple random walk on the integers: $X_0 = 0$ and $X_{t+1} = X_t \pm 1$, each possibility having probability 1/2. (The frog flips a coin at each step to decide whether to move right or left by one step.) Compute the probability that $X_{2n} = 0$ (in $2n$ steps the frog is back at the starting point). Give a simple closed-form expression.

(b) Asymptotically evaluate this probability. Your answer should be of the form an^bc^n. Determine the constants a, b, c.

19. (20+10 points) Consider a random graph G on a given set of n vertices. Let $\alpha(G)$ denote the size of the largest independent set of G.

    (a) Prove: for almost all graphs G we have $\alpha(G) \leq 1 + 2\log_2 n$.

    (b) Prove: for almost all graphs G we have $\chi(G) \geq (\alpha(\overline{G}))^{100}$. Here $\chi(G)$ denotes the chromatic number of G and $\overline{G}$ is the complement of G, so $\alpha(\overline{G})$ is the clique number of G.

20. (4 + 14 + 6 + 8 + 8 points) We have n guests and n gift items. For each gift item, we draw a guest's name at random. The same name can be drawn multiple times.

    (a) What is the size of the sample space for this experiment?

    (b) A guest is unlucky if his/her name is never drawn. Let X be the number of unlucky guests. Determine $E(X)$.

    (c) Asymptotically evaluate your answer to (b). Give a very simple expression.

    (d) Let p_n denote the probability that $X = 0$ (none of the guests is is unlucky). Detemine p_n (give a simple closed-form expression).

    (e) True or false: $p_n < 1/2.7^n$ for all sufficiently large n. Prove your answer. (Note: $\mathrm{e} = 2.718\ldots$)

21. (8 points) Let F_n denote the n-th Fibonacci number (starting with $F_0 = 0$, $F_1 = 1$). Prove: for all n, the numbers F_n and F_{n+2} are relatively prime.

22. (16 points) Find an integer x between 1 and 30 such that for every integer $a \geq 0$ we have $a^x \equiv a^{7^{150}} \pmod{31}$. (The exponent is 7^{150}.) Do not use a calculator.

23. (BONUS 8B points) Let $n = pq$ where p, q are distinct primes. Prove that the following statement is false:
    $(\forall a)(\text{if } \gcd(a, n) = 1 \text{ then } a^{n-1} \equiv 1 \pmod{n})$.

24. (BONUS: 15B points) Recall that the distance between a pair of vertices of a graph is the length of the shortest path between them. The

diameter of a graph is the maximum of the distances between pairs of vertices. Prove: almost all graphs on n vertices have diameter 2.

25. (BONUS: 4 points) Prove: if a 2×2 real matrix A has no real eigenvalues then A is diagonalizable over $\mathbb{C}$.

26. (BONUS: 8B points) We say that a square matrix $A \in M_n(\mathbb{R})$ is *diagonally dominated* if $(\forall i)(|\alpha_{ii}| > \sum_{j\neq i} |\alpha_{ij}|$. (Each diagonal entry is greater in absolute value than the sum of the absolute values of all other entries in the same row.) Suppose A is diagonally dominated. Prove: $\det(A) \neq 0$. (Hint: prove that the columns are linearly independent.)

27. (BONUS: 10B points) Let G be a connected, non-bipartite graph. Let $f(k)$ denote the number of closed walks of length k in G. Prove: $f(k) \sim \lambda_1^k$ where λ_1 is the largest eigenvalue of A.

28. (BONUS 15B points) Let $A_1, \ldots, A_m$ be events such that $(\forall i)(P(A_i) = 1/2)$ and $(\forall i \neq j)(P(A_i \cap A_j) \leq 1/5)$. Prove: $m \leq 6$.

29. (BONUS: 10B points) Prove: a planar graph G with $n \geq 3$ vertices has at most $3n - 6$ edges. Hint: we may assume G is connected (why?). Note that every region has at least 3 sides. Infer that $3r \leq 2m$ where r is the number of regions. State and use Euler's formula. Where did the argument use the assumption that $n \geq 3$?