CMSC-37110 Discrete Mathematics
MIDTERM EXAM November 15, 2012

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This exam contributes 20% to your course grade.

Do **not** *use book, notes.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else.

1. (7 points) For what values of the integer m is the following statement true:

$$(\forall x, y)(xy \equiv 0 \pmod m \Rightarrow (x \equiv 0 \pmod m \text{ or } y \equiv 0 \pmod m))$$

2. (14 points) Compute without any calculations the value x such that $0 \le x \le 30$ and $x \equiv 7^{30000000001} \pmod{31}$. Reason your answer.

3. (10+5 points) Let $a = 5k+3$ and $b = 9k-2$. (a) Prove: $\gcd(a,b)$ is either 1 or 37. (b) Find infinitely many values of k such that $\gcd(a,b) = 37$.

4. (15 points) Let p be an odd prime. Suppose for some x we have $x^2 \equiv -1 \pmod p$. Prove: $p \equiv 1 \pmod 4$. Hint: Fermat's little Theorem.

5. (5+8 points) (a) Define the Erdős - Rado arrow symbol $n \to (k, \ell)$. (b) Prove: $(\forall r, s \ge 1)(rs \not\to (r+1, s+1))$.

6. (6 points) True or false: $\ln n = \Theta(\log_2 n)$. Prove your answer.

7. (15 points) Prove: for all sufficiently large values of n, the number of non-isomorphic trees with n vertices is greater than 2.7^n.

8. (5 points) Evaluate this expression in closed form:

$$\sum_{k=0}^{n} \binom{n}{k} 2^{-k/2}.$$

9. (10 points) Construct a nonnegative random variable X such that $E(X) = 1$ and $\mathrm{Var}(X) = 10$. Make your sample space as small as possible.

10. (15 points) Prove: for all sufficiently large n, the probability that a random graph on a given set of n vertices is planar is less than $2^{-0.49n^2}$.

11. (4+2+10 points) Let G_n be a random graph on a given set of n vertices. Let X_n denote the number of triangles in G_n. (a) Determine $E(X_n)$. (b) What is the size of the sample space of this experiment? (c) In class we proved that $\mathrm{Var}(X_n) \sim n^4/128$. Prove the Weak Law of Large Numbers for X_n. This means the following. Given $\epsilon > 0$, let $p(n, \epsilon)$ denote the probability that $|X_n - E(X_n)| \geq \epsilon E(X_n)$. Prove: for every $\epsilon > 0$ we have $\lim_{n\to\infty} p(n, \epsilon) = 0$.

12. (16 points) Let $p_1, \ldots, p_n$ be distinct odd prime numbers, and let N denote their product. How many integers x among $0, 1, \ldots, N-1$ satisfy the congruence $x^2 \equiv 1 \pmod{N}$? Your answer should be a very simple formula. Prove your answer. Name the result used.

13. (5+9+6 points) Let X be an integer chosen at random from the set $\{0, 1, \ldots, n\}$. Let $Y = \binom{n}{X}$. Determine (a) $E(Y)$ and (b) $\mathrm{Var}(Y)$. Give simple closed-form expressions. (c) Show that $\mathrm{Var}(Y) \sim an^b c^n$ for some constants a, b, c; determine a, b, c.

14. (16 points) We have a deck of n cards, numbered $\{1, 2, \ldots, n\}$. A "hand" consists of k cards. We shuffle the cards and deal a hand. What is the probability that the hand does not include consecutive pairs of numbers? (i and $i+1$ are consecutive numbers.) Your answer should be a very simple closed-form expression (a quotient of two binomial coefficients, no summation, no dot-dot-dots). Prove your answer. State the size of the sample space of this experiment.

15. (7 points) Recall that $\pi(n)$ denotes the number of primes $\leq n$. Decide which of the following three statements, involving the little-oh notation, is true. State and prove your answer. Name and state the theorem used.
(a) $\pi(n) = o(n^{0.9})$ (b) $n^{0.9} = o(\pi(n))$ (c) neither.

16. (10+4B points) Prove: if $1 \leq k \leq n$ then $\binom{n}{k} < \left(\frac{en}{k}\right)^k$.
(b) (BONUS) Prove: $\sum_{j=0}^{k} \binom{n}{j} < \left(\frac{en}{k}\right)^k$.

17. (BONUS) (4B points) Prove that a triangle-free graph has chromatic number $O(\sqrt{n})$.

18. (BONUS) (4B points) Prove: for almost all graphs G we have $\chi(G) > (\alpha(\overline{G}))^{100}$.

19. (BONUS) (3B points) Prove: all longest paths in a tree share a vertex.

20. (BONUS) (4B points) Let G denote a random graph on a given set of n vertices. What is the probability that all vertices of G have even degree? Your answer should be a simple closed-form expression.

21. (BONUS) (4B points) Let $S(n, 3) = \sum_{k=0}^{\infty} \binom{n}{3k}$.
Prove: $|S(n, 3) - 2^n/3| < 1$.