CMSC-37110 Discrete Mathematics
SECOND QUIZ November 1, 2012

Name (print): ____

Do **not** *use book, notes, scratch paper.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else. **Write your solution in the space provided.** You may CONTINUE ON THE REVERSE. This exam contributes 6% to your course grade.

1. (1+9 points) (a) Count the strings of length n over the alphabet $\{A, B, C, D, E\}$. (b) How many among these strings use all the five letters? Your answer should be a closed-form expression. State, do not prove.

2. (3+7 points) (a) Count the triangles in the complete graph K_n. (b) Count the 4-cycles (cycles of length 4) in the complete bipartite graph $K_{r,s}$. (Two 4-cycles count as equal if they have the same set of edges.) Your answers should be simple closed-form expressions. State, do not prove.

3. (10 points)

   Let G be a graph with n vertices and m edges. Let us color the vertices red/blue at random. What is the expected number of edges that connect two vertices of the same color? Prove your answer; give a clear definition of your variables. What is the size of the sample space?

4. (4+7 points) (a) Prove: $k! \geq (k/\mathrm{e})^k$ for all $k \geq 0$. (b) Use part (a) to prove: $\ln k! \sim k \ln k$. Do not use Stirling's formula.

5. (2+8+3B+3B points) Let A be a set of n elements. Recall that a *relation* on A is a subset of $A \times A$. (a) Count all relations on A. Your answer should be a very simple closed-form expression. (b) Let $B(n)$ denote the number of equivalence relations on A. Prove: $B(n) \leq n^n$. (c) (BONUS) Prove: $(\forall k \geq 0)(B(n) \geq k^{n-k})$. (d) (BONUS) Prove: $\ln B(n) \sim n \ln n$.

6. (7+2 points) For $n \geq 1$, prove:

$$\frac{4^n}{2n+1} < \binom{2n}{n} < 4^n.$$

Do not use Stirling's formula. Hint: what is the sum $\sum_{i=0}^{2n} \binom{2n}{i}$?

7. (BONUS PROBLEM: 3B points) Let p be an odd prime. Suppose for some x we have $x^2 \equiv -1 \pmod{p}$. Prove: $p \equiv 1 \pmod{4}$.