CMSC-37110 Discrete Mathematics
THIRD QUIZ November 29, 2012

Name (print): ____________________________________

Do **not** *use book, notes, scratch paper.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else. **Write your solution in the space provided.** You may CONTINUE ON THE REVERSE. This exam contributes 6% to your course grade.

1. (14 points) True or false (circle one). If **"false," give a counterexample.** If "true," do not prove. (Lose 1 point per wrong or missing answer and 2 points per wrong or missing counterexample.)

   (a) If a list of vectors spans the space then they are linearly independent **T F**

   (b) If a list of vectors is linearly dependent then one of the vectors must be a scalar times another vector on the list **T F**

   (c) The empty list of vectors is linearly independent **T F**

   (d) A list consisting of one vector is always linearly independent **T F**

   (e) If a 3×3 matrix A has rank 1 then it must have a pair of equal entries **T F**

   (f) Zero cannot be an eigenvalue **T F**

   (g) If the eigenvectors v_1, v_2, v_3 of the 3×3 matrix B are linearly independent then the corresponding eigenvalues $\lambda_1, \lambda_2, \lambda_3$ must be distinct ($Bv_i = \lambda_i v_i$). **T F**

2. (4+4 points) Consider a horizontal unit vector e_1 and a vertical unit vector e_2 in the plane. (a) Describe in English the linear transformation σ which has the matrix

$$S = [\sigma]_{\{e_1,e_2\}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

with respect to the basis (e_1, e_2). Draw the vector $e_1 + e_2$ and its image $\sigma(e_1 + e_2)$. (b) Find all eigenvectors and eigenvalues of σ.

3. (6 points) Let A be the adjacency matrix of a directed graph in which all vertices have out-degree d. Prove that d is an eigenvalue of A; find an eigenvector.

4. (8 points) Let A be the adjacency matrix of an undirected graph with n vertices and m edges. Determine $\mathrm{Tr}(A^2)$. Your answer should be a very simple expression in terms of these basic parameters.

5. (3+3+3+3 points) Consider the 3×3 matrix

$$T = \begin{pmatrix} F_k & F_{k+1} & F_{k+2} \\ F_{k+1} & F_{k+2} & F_{k+3} \\ F_{k+2} & F_{k+3} & F_{k+4} \end{pmatrix}$$

where F_k is the k-th Fibonacci number. (a) Find a nontrivial linear relation between the columns of T (i.e., a nontrivial linear combination that gives the zero vector). (b) Find $\mathrm{rk}(T)$; reason your answer. (c) Find $\det(T)$; prove your answer. (d) Find an eigenvector to the eigenvalue 0; do not prove.
(This problem requires no calculation.)

6. (12 points) Let A, B be matrices such that AB is defined. Prove: $\mathrm{rk}(AB) \le \mathrm{rk}(A)$. Clarity matters.

7. (BONUS) (3B points) Compute the determinant of the $n \times n$ tridiagonal matrix B_n which has 1 in the diagonal, 1 immediately above the diagonal, and -1 immediately below the diagonal. The figure shows the case $n = 8$.

$$B_8 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

8. (BONUS) (4B points) Let A be the adjacency matrix of a bipartite (undirected) graph. Prove: if λ is an eigenvalue of A then $-\lambda$ is also an eigenvalue.

9. (BONUS) (1B points) A *right eigenvector* of a matrix $A \in M_n(\mathbb{R})$ is a nonzero column vector ($n \times 1$ matrix) x such that $Ax = \lambda x$ for some scalar λ. A *left eigenvector* is a row vector y ($1 \times n$ matrix) such that $yA = \mu y$. Prove: if $\lambda \ne \mu$ then x and y are orthogonal, i. e., $yx = 0$.

10. (BONUS) (4B points) Suppose $X_1, \ldots, X_k$ are pairwise independent non-constant random variables with expected value zero over a sample space of n elements. Prove: $n \ge k$. Hint: prove that $X_1, \ldots, X_k$ are linearly independent.

11. (BONUS) (8B points) Consider the $n \times n$ matrices $A = (\alpha_{ij})$ and $B = (\beta_{ij})$ where $\beta_{ij} = \alpha_{ij}^2$. Let $\mathrm{rk}(A) = k$. Prove: $\mathrm{rk}(B) \le \binom{k+1}{2}$.