CMSC-37110 Discrete Mathematics
FINAL EXAM December 10, 2013

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This exam contributes 35% to your course grade.

Do **not** *use book, notes.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else.

1. (4 points) Prove: if $0 \le k < n/2$ then $\binom{n}{k} \le \binom{n}{k+1}$.

2. (6 points) Suppose you recieve equal amounts of spam email and non-spam email. Further suppose the probability that a spam email contains the word "free" is $\frac{1}{3}$ and the probability that a non-spam e-mail contains the word "free" is $\frac{1}{30}$. Your software tells you that you recieved an e-mail that contains the word "free." What is the probability that the email is spam?

3. (18 points) Let p be a prime number and let $f(x) = 1+x+x^2+\cdots+x^{p-2}$. Prove: $(\forall x)(f(x) \equiv -1, 0 \text{ or } 1 \pmod p)$.

4. (6+14 points) (a) Count the increasing functions $f : [k] \to [n]$. (Recall the notation $[k] = \{1, \dots, k\}$.) (b) Count those functions $f : [k] \to [n]$ that satisfy $f(i+1) \ge f(i) + 2$ for every i. Your answers should be simple expressions involving binomial coefficients. Prove your answers.

5. (5+5+5 points) (a) Find a sequence a_n such that $\lim_{n\to\infty} a_n = 1$ but $\lim_{n\to\infty} a_n^n = \infty$. (b) Find a sequence b_n such that $\lim_{n\to\infty} b_n = \infty$ but $b_{n+1} \sim b_n$. (c) Prove: if $c_n > 1$ and $c_{n+1} = O(c_n)$ then $\ln c_n = O(n)$.

6. (3+5+2 points) Let I denote the $n \times n$ identity matrix and J the $n \times n$ all-ones matrix (every entry is 1). Let $A = I + J$.
   (a) For the case $n = 3$ write down in full detail the system $Ax = 0$ of homogeneous linear equations. Call the unknowns x_1, x_2, x_3. Using this notation, what does x mean?
   (b) For every n, prove: the columns of A are linearly independent.
   (c) What does item (b) say about the solutions of the system $Ax = 0$?

7. (5+3+5+3B+5B+4B points) Let $n \ge 3$ and let A be an $n \times n$ matrix with characteristic polynomial $f_A(t) = t^n - 3t + 2$.
   (a) Decide whether or not A is singular. Clearly say YES or NO. Prove your answer. (Recall that an $n \times n$ matrix is *singular* if $\det(A) = 0$.)
   (b) Prove that A has an eigenvalue that is an integer.

(c) Find the sum of all the n (complex) eigenvalues of A. Indicate the facts you use to obtain your answer.

(d) (BONUS) Prove: A is not a stochastic matrix.

(e) (BONUS) Prove: If $n \geq 4$ then A is diagonalizable.

(f) (BONUS) Prove: Disprove the conclusion of (c) when $n = 3$.

8. (6+4B+4B points) Let V be an n-dimensional euclidean space.

   (a) Let $v_1, \ldots, v_k$ be pairwise orthogonal non-zero vectors in V. Prove that they are linearly independent.

   (b) (BONUS) Prove that the functions $\cos(t), \cos(2t), \ldots, \cos(kt)$ are linearly independent.

   (b) (BONUS) Let U be a k-dimensional subspace of V. Let $U^\perp$ denote the set of those vectors that are orthogonal to all vectors in U. Note that $U^\perp$ is a subspace. (You don't need to prove this.) Prove: $\dim(U^\perp) = n - k$.

9. (6+4+5+5+4B points) Let A be a real symmetric $n \times n$ matrix. We say that A is *positive semidefinite* if $(\forall x \in \mathbb{R}^n)(x^\dagger A x \geq 0)$ (where $\dagger$ indicates transpose). We say that A is *positive definite* if $(\forall x \in \mathbb{R}^n)(x \neq 0 \Rightarrow x^\dagger A x > 0)$

   (a) Prove: Prove that the all-ones matrix J is positive semidefinite.

   (b) Prove: $I + J$ is positive definite.

   (c) Prove: If A is positive definite then A is non-singular.

   (d) Prove: if A is positive semidefinite then all eigenvalues of A are non-negative.

   (e) (BONUS) State and prove the converse of (d).

10. (BONUS: 5B points) Let $A \in M_n(\mathbb{R})$ be a symmetric real matrix. Prove that all of its eigenvalues (over $\mathbb{C}$) are real. Do not use the Spectral Theorem.

11. (6+8+5+6+6B points)

    (a) Draw the diagram of a weakly connected finite Markov Chain which has more than one stationary distribution. Give two stationary distributions for your Markov Chain. Use as few states as possible.

    (b) Recall that a finite Markov Chain is *irreducible* if its transition digraph is strongly connected. Draw the diagram of a reducible (not irreducible) finite Markov Chain with a unique stationary distribution. State the stationary distribution. Do not prove. Use as few states as possible.

(c) Define ergodicity of a finite Markov Chain. Define the terms used in the definition in terms of directed graph concepts.

(d) Draw the diagram of an irreducible but non-ergodic finite Markov Chain.

(e) (BONUS) Prove: the stationary distribution of an irreducible finite Markov Chain is unique. (Prove uniqueness only. Do not prove the existence of a stationary distribution.) Do not use the Frobenius-Perron Theorem.

12. (15+7 points)

(a) Let $A \in M_n(\mathbb{R})$ be a symmetric real matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $\rho = \max\{|\lambda_i| : 1 \leq i \leq n\}$. Prove: $(\forall v \in \mathbb{R}^n)(\|Av\| \leq \rho\|v\|)$. *Hint:* Recall that the Spectral Theorem says that A has an orthonormal eigenbasis. Represent v as a linear combination of this basis. Recall that $\|v\|^2 = v^\dagger v$.

(b) Find a real 2×2 matrix B with real eigenvalues and a vector $v \in \mathbb{R}^2$ such that $\|Bv\| > \rho\|v\|$. State the eigenvalues and the value ρ for your matrix. *Hint:* Make B triangular.

13. (5+15+5 points) We roll n dice; the numbers shown are $X_1, \ldots, X_n$. $(1 \leq X_i \leq 6.)$ Let $Y = \sum_{i=1}^{n-1} X_i X_{i+1}$. Compute (a) $E(Y)$ (b) $\mathrm{Var}(Y)$. (c) Asymptotically evaluate $\mathrm{Var}(Y)$. Your answers to each question should be simple closed-form expressions.

14. (3+4+4+5B+5B points) Let (Ω, P) be a probability space with $|\Omega| = n$.

(a) What do we call the elements of the function space $\mathbb{R}^\Omega$?

(b) What is the dimension of $\mathbb{R}^\Omega$? Describe a basis.

(c) Let $X_1, \ldots, X_k$ be pairwise independent random variables such that $(\forall i)(E(X_i) = 0$ and $\mathrm{Var}(X_i) = 1)$. For all i and j, determine $E(X_i X_j)$.

(d) (BONUS) Prove: under the assumptions of (c), the random variables $X_1, \ldots, X_k$ are linearly independent.

(e) (BONUS) Prove: If there exist k non-trivial, pairwise independent events in (Ω, P) then $n \geq k + 1$.

15. (5+6+10 points) (a) Draw a topological $K_{3,3}$ with 10 vertices. (b) State Kuratowski's characterization of planar graphs. (c) Prove: if a connected graph has n vertices and $n + 2$ edges then it is planar.

16. (18 points) Prove: for all sufficiently large n, the probability that a random graph is planar is less than $2^{-0.49n^2}$.

17. **(20 points)** Prove: almost all graphs on n vertices have no clique (complete subgraph) of size $\geq 1 + 2\log_2 n$. (Hint: estimate the probability of cliques of size k. Do not substitute the value $1 + 2\log_2 n$ for k until the very end to avoid messy formulas.)

18. **(8+12 points)** (a) State the multinomial theorem: express $(x_1 + \cdots + x_k)^n$ as a sum. Express the coefficients in terms of factorials. (b) Count the terms in your expression. Your answer should be a very simple expression (a binomial coefficient).

19. **(1+7 points)** (a) Define the little-oh notation.
(b) Prove: $n^{100} = o(1.01^n)$. Elegance counts. Do not use L'Hospital's rule beyond using the fact that $\lim_{x\to\infty} \ln x/x = 0$. (Hint: substitute a new variable.)

20. **(1+9 points)** (a) Count the strings of length n over the alphabet $\{A, B, C, D, E\}$. (b) How many among these strings use all the five letters? Your answer should be a closed-form expression.

21. **(8 points)** Construct a probability space and two random variables that are uncorrelated but not independent. Make your sample space as small as possible. (Recall: the random variables X, Y are *uncorrelated* if $E(XY) = E(X)E(Y)$.)

22. **(5+15 points)** (a) Prove: if p is a prime then the only solutions to the congruence $x^2 \equiv 1 \pmod{p}$ are $x \equiv \pm 1 \pmod{p}$. (b) Let $p < q < r$ be three distinct odd primes. Let $n = pqr$. Count the solutions to the congruence $x^2 \equiv 1 \pmod{n}$. (Two solutions count as distinct if they are not congruent modulo n.) Prove your answer.

23. **(10+10 points)** Let $a_n > 2$ and $b_n > 2$ be sequences of real numbers. Consider the following two statements: (1) $a_n = \Theta(b_n)$; (2) $\ln a_n \sim \ln b_n$. (a) Prove that (2) does not follow from (1). (b) Prove that if $a_n \to \infty$ then (2) follows from (1).

24. **(2+8 points)** Let X be a random nonnegative integer with 100 decimal digits; initial zeros are permitted. (Each of the 100 digits is chosen at random from $\{0, 1, \ldots, 9\}$.) (a) What is the size of the sample space of this experiment? (b) Estimate the probability that X is prime. Use the approximation $\ln 10 \approx 2.303$. Do not use a calculator. Your answer should be a simple fraction.

25. **(BONUS 6B points)** Let $n = pq$ where p, q are distinct primes. Prove that the following statement is false:
$(\forall a)(\text{if } \gcd(a, n) = 1 \text{ then } a^{n-1} \equiv 1 \pmod{n})$.