CMSC-37110 Discrete Mathematics
MIDTERM EXAM November 12, 2013

Instructor: László Babai Ryerson 164 e-mail: laci@cs

This exam contributes 20% to your course grade.

Do **not** *use book, notes.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else.

1. (6 points) True or false: $\ln n = \Theta(\log_2 n)$. Prove your answer.

2. (6 points) Find two sequences a_n and b_n of positive numbers such that $a_n = \Theta(b_n)$ but $\lim_{n\to\infty} a_n/b_n$ does not exist.

3. (16 points) Construct a nonnegative random variable X such that $E(X) = 1$ and $\mathrm{Var}(X) = 10$. Make your sample space as small as possible.

4. (5+5+15 points) Recall that the *generating function* of the sequence $(a_0, a_1, a_2, \dots)$ is the function $\sum_{n=0}^{\infty} a_n x^n$. Give closed-form expressions for the generating functions of the following sequences:

   (a) $a_0 = 1$, $a_n = 2a_{n-1}$,

   (b) $b_0 = 1$, $b_n = b_{n-1}/n$,

   (c) $c_0 = 1$, $c_n = (101 - n)c_{n-1}/n$.

5. (20 points) Determine the number of independent sets in the path of length n. Hint: Call this number $f(n)$. Determine $f(n)$ for small values of n. Observe the pattern. Prove. Your answer should be a closed-form expression in terms of a familiar sequence.

6. (2+8+8+3+6 points) Let $G = (V, E)$ be a graph with n vertices and m edges. Color each vertex red or blue at random (flipping a fair coin for each vertex). For each edge $e \in E$, let Y_e denote the indicator variable of the event that e is "legal," i. e., e joins a red vertex and a blue vertex.

   (a) What is the size of the sample space for this experiment?

   (b) Are the variables Y_e (b1) pairwise independent? (b2) independent?

   (c) Let X denote the number of legal edges. Determine (c1) $E(X)$ and (c2) $\mathrm{Var}(X)$.

7. (5+14+4 points) Let p be a prime number.

   (a) Prove: if $1 \le k \le p-1$ then $\binom{p}{k}$ is divisible by p. Explicitly use the prime property of p in your proof.

(b) Use part (a) to prove by induction on m that $(\forall m \geq 0)(m^p \equiv m \pmod{p})$.

(c) Use (b) to prove that $(\forall a)(a^p \equiv a \pmod{p})$.

Do not use Fermat's little Theorem; note that this sequence of problems gives a new proof of FlT.

8. (3+10+5B points)

   (a) Define the diameter of a graph.

   (b) Determine the minimum number of edges among graphs with n vertices and diameter 2.

   (c) (BONUS) Prove: for all sufficiently large n $\exists$ graph with n vertices, maximum degree $\leq n/100$, diameter 2, and $O(n)$ edges.

9. (10 points) Count the 4-cycles in the complete bipartite graph $K_{r,s}$. (Two 4-cycles count as equal if they have the same set of edges.) Your answer should be a simple closed-form expression.

10. (14 points) Recall that the *chromatic polynomial* $f_G(x)$ of the graph G is the polynomial which counts the legal colorings of G from the set $[x] = \{1, 2, \ldots, x\}$ of colors. (x is a positive integer. Not all colors need to be used.) Let T be a tree with n vertices. Prove: $f_T(x) = x(x-1)^{n-1}$.

11. (14 points) Recall that a graph is *bipartite* if it is 2-colorable. Prove: if a graph G has m edges then G has a bipartite subgraph with at least $m/2$ edges. Hint: Problem 6 (c1).

12. (4 + 2 + 20 points) In class we proved the following statement:
    (*) "Almost all graphs have diameter 2."

   (a) Explain the precise meaning of statement (*).

   (b) What is the size of the sample space for this experiment?

   (c) Prove statement (*).

13. (BONUS: 5B points) Prove: almost all graphs G satisfy the inequality

$$\chi(G) > (\alpha(\overline{G}))^{100}.$$

14. (BONUS: 6B+4B points) Let $r(G) = \max(\alpha(G), \alpha(\overline{G}))$. Prove:
    (a) $(\forall G)(r(G) = \Omega(\log n))$ (b) $(\forall n)(\exists G)(r(G) = O(\log n))$.

15. (BONUS: 5B points) Prove: all longest paths in a tree share a vertex.

16. (BONUS: 5B points) Prove: $\sum_{j=0}^{k} \binom{n}{j} < \left(\frac{en}{k}\right)^k$.