CMSC-37110 Discrete Mathematics
FIRST QUIZ October 17, 2013

Name (print): ________________________________

Do **not** *use book, notes, scratch paper.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else. **Write your solution in the space provided.** You may **continue on the reverse**. This exam contributes 8% to your course grade.

1. (3+18 points: 1 point for each answer, 6 points for each proof)
   True or **false** (circle one, prove). All quantifiers range over the integers.
   (a) $(\exists x)(\forall y, z)(x \neq 15y + 22z)$ **T** **F**
   (b) $(\forall x)(\exists y)(xy \equiv 1 \pmod 7 \text{ OR } (x+1)y \equiv 1 \pmod 7)$ **T** **F**
   (c) $(\forall x)(\exists y)(xy \equiv 1 \pmod 6 \text{ OR } (x+1)y \equiv 1 \pmod 6)$ **T** **F**

2. (6 points) Let $\{a_n\}$ be a sequences of real numbers. Define what it means that $\lim_{n\to\infty} a_n = \infty$. Your answer should be a properly quantified formula with no words.

3. (15 points) Let $a = 5k + 7$ and $b = 4k - 3$.
   Prove: $\gcd(a, b)$ is either 1 or 43.

4. (8 points) A relation R is *irreflexive* if $(\forall a)((a,a) \notin R)$. Let A be a set of n elements. What is the number of irreflexive relations on A ?

5. (3+15 points: 1 point for each answer, 5 points for each proof) **True** or **false** (circle one, prove). Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. Suppose $a_n \sim b_n$ and $\lim_{n\to\infty} a_n = \infty$. Does it follow that

   (a) $\lim_{n\to\infty} b_n = \infty$ **T** **F**

   (b) $(\exists c)(\forall n)(|a_n - b_n| \leq c)$ **T** **F**

   (c) $\ln(5a_n^2) \sim \ln(7b_n^2)$ **T** **F**

6. (12 points) Prove that the following system of simultaneous congruences has no solution.

$$\begin{aligned} x &\equiv 11 \pmod{21} \\ x &\equiv 5 \pmod{12} \\ x &\equiv 1 \pmod{14} \end{aligned}$$

7. (BONUS: 5B points) Prove: the congruence $x^2 \equiv -1 \pmod{79}$ has no solution. (Note: 79 is a prime.) Use results proved in class only.

8. (BONUS: 5B points) Suppose P is the product of k consecutive integers ($k \geq 1$). Prove: $k! \mid P$.

9. (BONUS: 7B points) Let $S(n,3) = \sum_{k=0}^{\infty} \binom{n}{3k}$.

   Prove: $|S(n,3) - 2^n/3| < 1$.