CMSC-37110 Discrete Mathematics
SECOND QUIZ   December 3, 2013

Name (print): ______________________________

Do **not** *use book, notes, scratch paper.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else. **Write your solution in the space provided.** You may **continue on the reverse**. This exam contributes **5%** to your course grade.

1. (5 points) Draw a strongly connected digraph with period 3 that has no directed cycle of length 3. Use as few edges as possible; state the number of edges you use.

2. (6 points) Prove: if a finite Markov Chain has two stationary distributions then it has infinitely many.

3. (15 points: 1 point for each answer, 3+4+5 points for the proofs)
True or **false** (circle one, prove). (a) If two vectors are linearly dependent then one of them is a scalar times the other. **T** **F**
(b) If the determinant of a 3×3 matrix is zero then one of the columns is a scalar times another column. **T** **F**
(c) If A and B are 3×3 matrices with positive entries then
. $\operatorname{rk}(A + B) \geq \operatorname{rk}(A)$. **T** **F**

4. (7 points) Find the characteristic polynomial and the eigenvalues of the matrix $B = \begin{pmatrix} 5 & 2 \\ 4 & 7 \end{pmatrix}$.

5. (7 points) Find an $n \times n$ matrix A such that $\text{rk}(A) = 1$ and all the n^2 entries of A are distinct.

6. (10 points) Let $\alpha_1, \dots, \alpha_n$ be distinct real numbers. Let $f(x) = \prod_{i=1}^{n}(x-\alpha_i)$ and let $g_i(x) = f(x)/(x-\alpha_i)$. (So each g_i is a polynomial of degree $n-1$.) Prove that $g_1, \dots, g_n$ are linearly independent in the space $\mathbb{R}[x]$ of polynomials over $\mathbb{R}$.

7. (BONUS: 4B points) Let A be an $n \times n$ matrix with integer entries. Suppose all diagonal entries are odd and all other entries are even. Prove that A is non-singular (i.e., $\det(A) \neq 0$).

8. (BONUS: 4B points) Prove: every non-singular $n \times n$ matrix can be turned into a singular matrix by changing just one entry.

9. (BONUS: 2B points) Let A, B be $n \times n$ stochastic matrices. Prove: $A - B$ is singular.