The “Greedy matching” problem

A matching in a graph $G = (V, E)$ is a set $M \subseteq E$ of pairwise disjoint edges. The size of a matching is the number of edges in $M$. The matching number $\nu(G)$ is the maximum size of a matching in $G$. For instance, $\nu(K_n) = \lceil n/2 \rceil$ and $\nu(K_{r,s}) = \min\{r, s\}$ (verify these to make sure you understand the definition!).

A matching $M$ is maximal if it cannot be extended, i.e., if there is no matching that properly contains $M$.

A matching $M$ is maximum if $|M| = \nu(G)$.

Note that every maximum matching is maximal but not conversely (verify!).

We wish to estimate $\nu(G)$ using a greedy approach. It will turn out that the worst error we can make in doing so is a factor of 2.

We assume $V = \{1, 2, \ldots, n\}$ and $E$ is given as a list $e_1, e_2, \ldots, e_m$.

The greedy matching algorithm is described by the following pseudocode:

```
0 Initialize: $M := \emptyset$
1 for $i = 1$ to $m$ do
2     if $e_i$ does not intersect any edge in $M$ then add $e_i$ to $M$
3 end(for)
5 return $M$
```

It should be clear that the algorithm returns a matching (why?).

**Problem.** (a) Let $M$ be the matching returned by the greedy matching algorithm. Prove:
(a1) $M$ is a maximal matching.
(a2) $|M| \leq \nu(G) \leq 2|M|$.  \hspace{1cm} (1)

(b) Prove that the upper bound is tight in the following sense:
\[ (\forall k \geq 0)(\exists G = (V, E) and an ordering of E)(\nu(G) = 2k and |M| = k). \]