CMSC-27410/37200 Honors Combinatorics
FINAL TEST June 4, 2014

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This exam contributes 18% to your course grade. Take this problem sheet home as a souvenir.

Do **not** *use book, notes.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the proctor.** The *bonus problems* are underrated, do not work on them until you are done with everything else.

1. (20 points) Prove: $(\exists c > 1)$ such that for all sufficiently large n there is a graph with n vertices which has at least c^n maximum independent sets. Make c as large as you can. State the value of c and the maximum size of independent sets in your graph.

2. (28 points) (Eventown Theorem, Berlekamp) Let $C_1, \dots, C_m$ be distinct subsets of $[n]$. Suppose $(\forall i, j)(|C_i \cap C_j|$ is even). Prove: $m \le 2^{\lfloor n/2 \rfloor}$.

3. (15 points) Recall that the Kővári - Sós - Turán Theorem states: If a graph G with n vertices and m edges has no 4-cycle then $m = O(n^{3/2})$. Prove that this result is tight up to a constant factor, i.e., there exists $c > 0$ such that for all $n \ge 2$, there is a C_4-free graph with n verices and $\ge cn^{3/2}$ edges.

4. (BONUS: 10B+5B points) Recall Chernoff's bound: if $X_1, \dots, X_n$ are independent random variables satisfying $E(X_i) = 0$ and $|X_i| \le 1$ for every i and $Y = \sum_{i=1}^n X_i$ then for every $a > 0$ we have $P(|Y| \ge a) \le 2\mathrm{e}^{-a^2/(2n)}$. (a) Prove Chernoff's bound for the special case when each X_i takes the values 1 and -1 only. (b) Prove the general case.

5. (20 points) Notation: $\exp(t) = \mathrm{e}^t$. Use Chernoff's bound to prove:

   Let us roll n dice; let Z denote the number of times we rolled a "6." Prove: for every $\epsilon > 0$ the probability that $P(|Z - n/6| \ge \epsilon n) \le 2\exp(-c\epsilon^2 n)$ for some constant $c > 0$. Make your constant c as large as you can; state the value of c you get.

6. (10+18 points)

   (a) Define the Shannon capacity $\Theta(G)$ of a graph G as the limit of certain sequence associated with G. Define the graph-product concept involved in the definition.

   (b) Prove: if G is self-complementary (isomorphic to its complement) then $\Theta(G) \ge \sqrt{n}$, where n is the number of vertices of G.

7. (5+8+20+10B points) Recall the definition of $\chi^*(G)$, the fractional chromatic number of the graph G: Let $[n]$ be the set of vertices of G and let $C_1, \dots, C_m$ denote the independent sets in G. Let x_i be real variables ($i \in [m]$) satisfying the following inequalities: for every $j \in [n]$, let s_j denote the sum of those x_i for which $j \in C_i$. Suppose $s_j \ge 1$ for every $j \in [n]$. Let $\chi^*(G)$ be the minimum of $\sum_{i=1}^m x_i$ under these constraints.

   (a) Prove: $\chi^*(G) \le \chi(G)$.

   (b) Compute $\chi^*(C_5)$. Prove your answer.

   (c) Prove: $\alpha(G)\chi^*(G) \ge n$

   (d) (BONUS) Prove: $\Theta(G) \le \chi^*(\overline{G})$.

8. (4+12 points) (a) Define Steiner Triple Systems (STS). (b) Prove: if there exists a STS with n points then there exists a STS with $3n$ points.

9. (BONUS: 10B points) Prove Sperner's Theorem: If $A_1, \dots, A_m$ are pairwise incomparable subsets of $[n]$ then $m \le \binom{n}{\lfloor n/2 \rfloor}$.

10. (20 points) (Littlewood–Offord problem) Let $a_1, \dots, a_n, b$ be real numbers, $a_i \ne 0$. Let I be a random subset of $[n]$. Prove: $P(\sum_{i \in I} a_i = b) \le c/\sqrt{n}$ for some constant $c > 0$. State the value of c you get for large n.

11. (BONUS: 10B points) Prove: If the graph G contains no 5-cycle then its chromatic number is $O(\sqrt{n})$ (where n is the number of vertices).

12. (BONUS: 8B points) Outline the proof of the following: For every g, k there exists a graph G of girth $\ge g$ and fractional chromatic number $\chi^*(G) \ge k$.

Total 180 points + 53 bonus points