Honors Combinatorics CMSC-27410/Math-28400/CMSC-37200 Sixth Quiz. May 21, 2014
Instructor: László Babai

Name (print): ____________________________ **Major/Year** ____________________

Show all your work. **Do not use book, notes, or scrap paper.** Write your answers in the space provided. You may **continue on the reverse.** This quiz contributes 5% to your course grade.

1. (7 points) Prove: If A is an $n \times n$ stochastic matrix then $\operatorname{per} A \leq 1$. ($\operatorname{per} A$ denotes the permanent of A.)

2. (3+18+6 points) Let $a_1, \ldots, a_n, b$ be real numbers; assume none of the a_i is zero. Choose a subset $I \subseteq [n]$ at random. (a) State the size of the sample space for this experiment.
(b) Prove: $P\left(\sum_{i \in I} a_i = b\right) = O(1/\sqrt{n})$. (c) Show that the $O(1/\sqrt{n})$ bound is best possible (i. e., that $o(1/\sqrt{n})$ is not a valid upper bound). Hint for (b): Use Sperner's Theorem that says that the maximum number of pairwise incomparable subsets of $[n]$ is $\binom{n}{\lfloor n/2 \rfloor}$. Two sets are *incomparable* if neither is a subset of the other.

3. (16 points) Let m be the number of edges and t the number of triangles in a graph. Prove: $t \leq cm^{3/2}$ where $c = \sqrt{2}/3$. Use without proof that if $x_1, \ldots, x_n$ are real numbers then $\left(\sum_{i=1^n} x_i^2\right)^3 \geq \left(\sum_{i=1^n} x_i^3\right)^2$.

4. (BONUS: 3 points) Let G be a graph with average degree d and adjacency matrix A. Let λ_1 be the largest eigenvalue of A. Prove: $\lambda_1 \geq d$.

5. (BONUS: 8 points) Construct 3 orthogonal Latin squares of order 4. (Your answer should be three 4×4 matrices.)