CMSC-37110 Discrete Mathematics
FINAL EXAM December 11, 2014

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This exam contributes 35% to your course grade.

Do **not** *use books, notes, electronic devices.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else.

1. (20+20+10B points) Let A be an $n \times n$ matrix and λ an eigenvalue. Recall: the *geometric multiplicity* of λ is the maximum number of linearly independent eigenvectors to eigenvalue λ. The *algebraic multiplicity* of λ is k if the characteristic polynomial $f_A(t)$ is divisible by the polynomial $(t-\lambda)^k$ but not divisible by the polynomial $(t-\lambda)^{k+1}$.

   (a) Determine (a1) the geometric and (a2) the algebraic multiplicity of the eigenvalue $\lambda = 2$ for the matrix

   $$A = \begin{pmatrix} 17 & 13 & 11 & 7 & 5 \\ 0 & 2 & 1 & -7 & 5 \\ 0 & 0 & 2 & 3 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 3 \end{pmatrix}$$

   (Hint: Use the Rank-Nullity Theorem to determine the geometric multiplicity.) **Show your work.** Result without details of the calculation will not be accepted. **DO NOT USE electronic devices.**

   (b) (BONUS) Prove: For any $n \times n$ matrix B and any eigenvalue λ of B, the geometric multiplicity of λ is always $\le$ the algebraic multiplicity.

2. (5+15+20+5 points) For each of the following relations on the universe specified, determine whether or not it is an equivalence relation. Clearly answer YES or NO. Prove your answers.

   (a) Universe: the integers ≥ 2. Relation: "not relatively prime."

   (b) Universe: all integers. Relation: "$x^2 \equiv y^{14} \pmod 7$."

   (c) Universe: all non-trivial events in the uniform probability space over a sample space of size n. Relation: "not independent." Your answer should depend on n.

   (d) Universe: all infinite sequences of real numbers. Relation: "$a_n - b_n = O(n)$."

3. (8+20+10 points) Let A be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $\mu = \max_i |\lambda_i|$.

   (a) The Spectral Theorem says that A has an orthonormal eigenbasis. What does this mean? Define "orthonormal eigenbasis." State how many vectors are in an orthonormal eigenbasis of A.

   (b) Prove: For every $v \in \mathbb{R}^n$, we have $\|Av\| \leq \mu\|v\|$. (Use the Spectral Theorem but no other theorems that we did not prove in class. Recall that the norm of $v = (v_1, \ldots, v_n)^T \in \mathbb{R}^n$ is defined as $\|v\| = \sqrt{v^T v} = \sqrt{\sum v_i^2}$. (^{T} stands for transpose).)

   (c) Prove: There exists $w \in \mathbb{R}^n$, $w \neq 0$ such that $\|Aw\| = \mu\|w\|$.

4. (16 points) A graph G has n vertices, out of which k vertices have degree 7 and the remaining $(n-k)$ vertices have degree 16. Determine the number of paths of length 2 in G. (Note: the path $u - v - w$ and the path $w - v - u$ count as the same path, so for instance K_3 has 3 paths of length 2.)

5. (9+9 points) Let $\varphi : V \to W$ be a linear map. For each of the following statements, decide whether or not the statement is true for every φ.

   (a) If the vectors $v_1, \ldots, v_k \in V$ are linearly independent then their images, $\varphi(v_1), \ldots, \varphi(v_k) \in W$ are linearly independent.

   (b) If the vectors $v_1, \ldots, v_k \in V$ are linearly dependent then their images, $\varphi(v_1), \ldots, \varphi(v_k) \in W$ are linearly dependent.

   If your answer is "YES," prove. If your answer is "NO," give a **specific counterexample** (define V, W, φ, and the vectors $v_1, \ldots, v_k$).

6. (12+15+15+6B points) Consider the following random walk on the number line. X_t denotes the position of our wandering particle at time t. We start at the origin: $X_0 = 0$; and then at each time step, the particle moves one step to the right with probability $2/3$ or one step to the left with probability $1/3$. Formally: $P(X_{t+1} = j+1 \mid X_t = j) = 2/3$ and $P(X_{t+1} = j-1 \mid X_t = j) = 1/3$.

   (i) Determine $E(X_t)$.

   (ii) Determine $\mathrm{Var}(X_t)$.

   (iii) Determine the probability $p_t(j) = P(X_t = j)$. Your answer should be a simple closed-form expression.

   (iv) (BONUS) What is the most likely position of the particle at time t? Call this position j_t, so $p_t(j_t) = \max_j p_t(j)$. Prove: $|j_t - E(X_t)| \leq 1$.

7. (3+15 points) Select a random integer X with n digits such that all digits are odd (i.e., the digits are from the set $\{1, 3, 5, 7, 9\}$).

(a) What is the size of the sample space for this experiment?

(b) What is the probability that each of the 5 odd digits actually occur in X? Your answer should be a closed-form expression (no summation symbols or dot-dot-dots).

8. (20 + 5 points) Let G be a graph with n vertices and $\le n$ edges.

   (a) Prove: $\chi(G) = O(\sqrt{n})$. ("χ" denotes the chromatic number.)

   (b) Prove that this bound is tight, i.e., construct an infinite family of graphs satisfying the condition and having chromatic number $\chi(G) = \Omega(\sqrt{n})$.

9. (2+12 points) We flip 5 fair coins. Let X_i be the indicator variable of the event that the i-th coin comes up "Heads." (a) State the size of the sample space for this experiment. (b) What is the probability of the event that $X_1 = X_2X_3 + X_4X_5$? **Show all your work.**

10. (20 points) Consider a Bernoulli trial with probability p of success, i.e., we flip a biased coin that comes up Heads with probability p and Tails with probability $1 - p$; "Heads" counts as "success." We keep flipping the coin until the first success. Let X denote the number of times we flipped the coin. Determine $E(X)$.

11. (18 points; lose up to 4 points for each mistake) Recall that an $n \times n$ matrix A is non-singular if an only if the columns of A are linearly independent. State five conditions that are equivalent to this: "A is non-singular if and only if . . . ". Complete the sentence with a statement involving the concept in parentheses in each case.

    (a) (determinant)

    (b) (rank)

    (c) (solutions to system of linear equations [which system?])

    (d) (eigenvalues)

    (e) (inverse)

12. (12 points) Let $A = (a_{ij})$ be an $n \times n$ matrix with integer entries. Assume each diagonal entry a_{ii} is odd and each off-diagonal entry a_{ij} $(j \ne i)$ is even. Prove: A is nonsingular.

13. (12 points) Asymptotically evaluate the binomial coefficient $\binom{3n}{n}$. Your answer should be of the form $\binom{3n}{n} \sim an^b c^n$ where a, b, c are constants. Determine a, b, c.

14. (12 points) Let $G = (V, E)$ and $H = (V, F)$ be two graphs with the same vertex set, V. Let $L = (V, E \cup F)$. Prove: $\chi(L) \le \chi(G)\chi(H)$. ("$\chi$" denotes the chromatic number.)

15. (12 points) Prove: if a finite Markov Chain has two stationary distributions then it has infinitely many.

16. (8 points) Evaluate this expression in closed form:

$$\sum_{k=0}^{n} \binom{n}{k} 2^{-k/2}.$$

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17. (BONUS 8 points) Recall: an $n \times n$ matrix is *stochastic* if it is the transition matrix of a finite Markov Chain. Prove: if λ is a (real or complex) eigenvalue of a stochastic matrix then $|\lambda| \leq 1$. (If you are uncomfortable with complex numbers, assume λ is real for 8 points.)

18. (BONUS 1+3+5 points) Let $A = (a_{ij})$ be the adjacency matrix of the graph $G = (V, E)$ where $V = [n] = \{1, \ldots, n\}$, so $a_{ij} = 1$ if the vertices i and j are adjacent and 0 otherwise. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of A. Express (a) $\sum_i \lambda_i$ (b) $\sum_i \lambda_i^2$ (c) $\sum_i \lambda_i^3$ in terms of simple combinatorial parameters of G such as the number of certain small subgraphs.

19. (BONUS 6B points) Prove:

$$\sum_{i=0}^{k} \binom{n}{i} \leq \left(\frac{ne}{k}\right)^k .$$

20. (BONUS 3B points) A graph is *self-complementary* if it is isomorphic to its complement. Prove: if G is self-complementary then $\chi(G) \geq \sqrt{n}$. (n is the number of vertices.)

21. (BONUS 3B points) Prove that the $3 \times 3 \times 3$ grid graph has no Hamilton path ending in the center. (A Hamilton path is a path that includes every vertex.) (The graph in question is a 3-dimensional grid; it has 27 vertices.)

22. (BONUS 2B points) Let $A, B \in M_n(\mathbb{R})$ (real $n \times n$ matrices). Prove: $AB - BA \neq I$ (where I is the identity matrix).