CMSC 37110 Homework #15

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14.3. Let x and y be vertices belonging to the same strong component of G. Then there is an $x \to y$ walk, say of length k_1, and a $y \to x$ walk, say of length k_2. Then there is a closed $x \to y \to x$ walk of length $k = k_1 + k_2$. Now, any closed walk passing through y (say of length m) can be extended to a closed walk passing through x by traversing $x \to y$, then traversing the closed $y \to y$ walk, then traversing $y \to x$. The length of this walk is $k_1 + m + k_2 = k + m$. Since this is a closed walk passing through x, it follows that $\operatorname{per}(x) \mid k + m$. But since the $x \to y \to x$ walk is also a closed walk passing through x, we have $\operatorname{per}(x) \mid k$. Hence $\operatorname{per}(x) \mid (k + m - k)$, i.e., $\operatorname{per}(x) \mid m$. So $\operatorname{per}(x)$ divides the lengths of every closed walk through y, hence their gcd. That is, $\operatorname{per}(x) \mid \operatorname{per}(y)$. But by the same argument, $\operatorname{per}(y) \mid \operatorname{per}(x)$, so because $\operatorname{per}(x)$ and $\operatorname{per}(y)$ are both positive, it follows that they are equal, that is, $\operatorname{per}(x) = \operatorname{per}(y)$ if x and y are in the same strong component.

□

15.3. The characteristic polynomial of R_θ is:

$$\begin{aligned} f_{R_\theta}(\lambda) &= \det \begin{pmatrix} \lambda - \cos\theta & \sin\theta \\ -\sin\theta & \lambda - \cos\theta \end{pmatrix} \\ &= (\lambda - \cos\theta)^2 + \sin^2\theta \\ &= \lambda^2 - 2\lambda\cos\theta + \cos^2\theta + \sin^2\theta \\ &= \lambda^2 - 2\lambda\cos\theta + 1. \end{aligned}$$

Let us calculate the roots. We can apply the quadratic formula to the characteristic polynomial, or get the solution faster by directly solving the equation $(\lambda - \cos\theta)^2 + \sin^2\theta = 0$, or, equivalently, $(\lambda - \cos\theta)^2 = -\sin^2\theta$, i.e., $\lambda - \cos\theta = \pm i\sin\theta$ where $i^2 = -1$. So the eigenvalues are $\lambda = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$. To find the eigenvectors, we have

$$\begin{aligned} \begin{pmatrix} (\cos\theta \pm i\sin\theta)\, x_1 \\ (\cos\theta \pm i\sin\theta)\, x_2 \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{pmatrix} \end{aligned}$$

So then

$$(\cos\theta \pm i\sin\theta)\, x_1 = x_1\cos\theta - x_2\sin\theta$$

$$\pm ix_1 \sin\theta = -x_2 \sin\theta$$
$$x_1 = \pm ix_2$$

so that eigenvector corresponding to $e^{\pm i\theta}$ is $\begin{pmatrix} \pm i \\ 1 \end{pmatrix}$.

□

15.4. *(a)* The characteristic polynomial of A is

$$f_A(\lambda) = \det \begin{pmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{pmatrix} = (\lambda - 1)^2 - (-1) \cdot 0 = (\lambda - 1)^2$$

which has a root of $\lambda = 1$, so the only eigenvalue of A is 1. To compute its eigenvectors, we solve $Ax = x$. We have

$$Ax = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 \end{pmatrix}$$

so we require $x_2 = 0$. That is, all eigenvectors of A are scalar multiples of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Since all eigenvectors of A are scalar multiples of each other, the space spanned by the eigenvectors is a one-dimensional subspace of $\mathbb{F}^2$. In particular, no two eigenvectors of A can span $\mathbb{F}^2$, so A does not have an eigenbasis.

(b) The characteristic polynomial of B is

$$f_B(\lambda) = \det \begin{pmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 2 \end{pmatrix} = (\lambda - 1)(\lambda - 2) - (-1) \cdot 0 = (\lambda - 1)(\lambda - 2)$$

which has roots (eigenvalues) $\lambda = 1, 2$. Since the eigenvalues are distinct, their corresponding eigenvectors are linearly independent, and thus are an eigenbasis because any two linearly independent vectors in $\mathbb{F}^2$ are a basis.

□

15.5. *(a)* First add each of the first $(n-1)$ rows to the last, so that every entry of the last row is $a + (n-1)b$. Then, subtract the last column from each of the remaining columns. This produces an upper triangular matrix where every diagonal entry (except that on the bottom row) is $a - b$ and the bottom-right entry is $a + (n-1)b$. The determinant of the matrix is then the product of the diagonal entries, that is, $(a - b)^{n-1}(a + (n-1)b)$.

(b1) If J is the $n \times n$ all-ones matrix, then we can write $J - \lambda I$ in the form above with $b = 1$ and $a = 1 - \lambda$. Then the characteristic polynomial is

$$f_J = (1 - \lambda - 1)^{n-1} (1 - \lambda + (n-1)1) = (-\lambda)^{n-1}(n - \lambda) = (-1)^n \left(\lambda^n - n\lambda^{n-1}\right)$$

(b2) Since the characteristic polynomial is $(-\lambda)^{n-1}(n - \lambda)$, it is easy to see that it has roots (eigenvalues) $\lambda = 0$ with multiplicity $n - 1$ and $\lambda = n$ with multiplicity 1.

(b3) [This solution was slightly modified by the instructor.] We note that

$$\begin{pmatrix}1 & 1 & \cdots & 1\\ 1 & 1 & \cdots & 1\\ \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & \cdots & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix} = \begin{pmatrix}x_1 + \cdots + x_n\\ x_1 + \cdots + x_n\\ \vdots\\ x_1 + \cdots + x_n\end{pmatrix}$$

So if $Jx = nx$ then all coordinates of nx must be the same; by scaling we may then choose $x_1 = \cdots = x_n = 1$, so let us take $v_1 = \begin{pmatrix}1\\ 1\\ \vdots\\ 1\end{pmatrix}$; this is an eigenvector to eigenvalue n. Let now $U = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 0\}$. So the nonzero vectors in the subspace U are precisely the eigenvectors of J to eigenvalue 0. For $i \geq 2$, let us choose $v_i \in U$ to have first coordinate 1 and i-th coordinate -1, all other coordinates 0. So the vectors v_i are the columns of the matrix

$$B = \begin{pmatrix}1 & 1 & 1 & \cdots & 1\\ 1 & -1 & 0 & \cdots & 0\\ 1 & 0 & -1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 0 & 0 & \cdots & -1\end{pmatrix}$$

We claim that $v_1, \ldots, v_n$ are linearly independent, i.e., the matrix B is nonsingular. Columns 2 to n are linearly independent because these columns contain the negative of the $(n-1) \times (n-1)$ identity matrix as a submatrix. So we just have to show that $v_1 \notin \text{span}\{v_i \mid 2 \leq i \leq n\}$. This is true because $v_2, \ldots, v_n \in U$ and $v_1 \notin U$. (Alternatively, we can prove that B is nonsingular by proving that $\det(B) \neq 0$. By adding rows $2, \ldots, n$ to the first row we get a triangular matrix with nonzero diagonal, so $\det(B) \neq 0$; in fact, $\det(B) = n(-1)^{n-1}$.)

We conclude that the vectors v_i are are linearly independent, hence a basis of $\mathbb{R}^n$, i.e., an eigenbasis of J.

□