CMSC-37110 Discrete Mathematics
MIDTERM EXAM November 18, 2014

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This exam contributes 20% to your course grade.

Do **not** *use book, notes.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else.

1. (10+10 points) True or false: (a) $n^2/4 = \Theta\left(\binom{n}{2}\right)$; (b) $2^{n^2/4} = \Theta\left(2^{\binom{n}{2}}\right)$. (Here the binomial coefficient is in the exponent.) Prove your answers.

2. (12 points) Find two sequences a_n and b_n of positive numbers such that $a_n = \Theta(b_n)$ but $\lim_{n\to\infty} a_n/b_n$ does not exist.

3. (9+9+9 points) Let G be a random graph on vertex set $[n] = \{1, 2, \dots, n\}$ where $n \geq 2$. Let X_i denote the degree of vertex i. (a) Are X_1 and X_2 independent? (b) Are X_1 and X_2 independent, given that vertices 1 and 2 are adjacent? (c) Are X_1 and X_2 independent, given that vertices 1 and 2 are not adjacent? Prove your answers.

4. (4+10+10+16+16+7 points) We have k processes and n memory cells; each process attempts to write independently into a randomly chosen memory cell. We say processes i and j *collide* if they try to write in the same cell. Let X denote the number of collisions, i. e., the number of pairs $\{i, j\}$ that collide. (So $0 \leq X \leq \binom{k}{2}$.) **(a)** What is the size of the sample space for this experiment? **(b)** Determine $E(X)$. Give a clear definition of the random variables you use. **(c)** Let $p(n, k)$ denote the probability that $X = 0$ (no collisions). Give a simple exact formula for $p(n, k)$. **(d)** Prove: If $k_1, k_2, \dots$ is a sequence of integers such that **(d1)** $\sqrt{n} = o(k_n)$ then $\lim_{n\to\infty} p(n, k_n) = 0$; **(d2)** $k_n = o(\sqrt{n})$ then $\lim_{n\to\infty} p(n, k_n) = 1$. **(e)** Explain the little-oh notation in parts (d1) and (d2) (state a limit relation in each case).

5. (5+25 points) Let $G = (V, E)$ be a bipartite graph. Color each vertex red or blue at random (flipping a fair coin for each vertex). **(a)** What is the size of the sample space for this experiment? **(b)** What is the probability that the coloring is legal? Express your answers in terms of simple parameters of the graph G. Explain your notation.

6. (30 points) Consider the vector space $V = \mathbb{R}^{\mathbb{N}}$ of infinite sequences $a = (a_0, a_1, a_2, \dots)$ (where $a_n \in \mathbb{R}$ and $\mathbb{N} = \{0, 1, 2, \dots\}$). Let us say that such a sequence is *Fibonacci-type* if $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 2$. Let W denote the set of Fibonacci-type sequences. Observe but do not prove that W is a subspace of V. Consider the linear transformation φ

on W that shifts the sequence to the left: $(\varphi a)_n = a_{n+1}$. For instance, $\varphi(3, 4, 7, 11, 18, \dots) = (4, 7, 11, 18, 29, \dots)$. Observe but do not prove that this is a linear transformation of W. Find all eigenvalues and eigenvectors of φ on W.

7. (8+10 points) For a graph G, let $\beta(G)$ denote the size of the smallest maximal independent set. ("Maximal" means you cannot add any vertex to it.) (a) Find a graph with n vertices and the largest possible ratio $\alpha(G)/\beta(G)$ (where, as always, $\alpha(G)$ denotes the independence number of G, i. e., the size of the largest independent set in G.) (b) Determine $\alpha(C_n)$ and $\beta(C_n)$ where C_n is the cycle of length n.

8. (BONUS: 10 points) [Note: this problem as stated in the midterm had an error. This is the updated version.]
   Let A be an $n \times n$ matrix with entry $a_{ij} \in \mathbb{R}$ in position (i, j). Assume the columns of A are linearly independent. Prove: one can always change the value of an entry in the first row such that the columns of A become linearly dependent. – Use the following result: every list of $k + 1$ vectors in $\mathbb{R}^k$ is linearly dependent.

9. (BONUS: 10 points) Let G be a regular graph of degree $d \geq 1$ (every vertex has degree d) with n vertices. Prove: $\alpha(G) \leq n/2$.

10. (BONUS: 12 points) Let $A \subseteq \mathbb{Z}$ be a set of n numbers. Let $B = A + \dots + A$ where the sum has k terms. Prove:

$$|B| \leq \binom{n+k-1}{k}.$$

11. (BONUS: 12 points) Let $X_1, \dots, X_k$ be random variables with expected value zero and variance 1. Prove: if the X_i are pairwise independent then they are linearly independent (as functions in the function space $\mathbb{R}^\Omega$).