CMSC-37110 Discrete Mathematics
FIRST QUIZ October 16, 2014

Name (print): ______________________________

Do **not** *use book, notes, scratch paper.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else. **Write your solution in the space provided.** You may **continue on the reverse**. This exam contributes 8% to your course grade.

1. (8 points) Is $6\mathbb{Z} \cup 7\mathbb{Z} \cup 8\mathbb{Z}$ a subgroup of $\mathbb{Z}$?
   Prove your answer. (Recall: a subgroup of $\mathbb{Z}$ is a non-empty subset of $\mathbb{Z}$ that is closed under subtraction.)

2. (16 points) Find a relation R on a small set A such that
   (a) R is NOT symmetric but
   (b) the transitive closure T of R is symmetric.
   Make A as small as possible. Indicate why (a) and (b) hold.

3. (5 points) Name the largest number M such that the following is true. Briefly reason your answer.
   $(\forall a, b)\,(((a \equiv b \pmod{28}) \wedge (a \equiv b \pmod{35})) \Rightarrow (a \equiv b \pmod{M}))$

4. (15 points; lose 7 points for each mistake in your list) List all pairs (x, y) of positive integers such that $\operatorname{lcm}(x, y) = \gcd(x, y) + 10$ and $x \le y$. Prove you did not miss any solutions.

5. (14 points) Prove that the following system of simultaneous congruences has no solution. Elegance counts.

$$\begin{aligned} 3x &\equiv 1 \pmod{20} \\ x &\equiv 32 \pmod{55} \\ x &\equiv 21 \pmod{44} \end{aligned}$$

6. (22 points) Calculate $19^{111111} \bmod 88$. (Your answer should be an integer between 0 and 87.) SHOW ALL YOUR WORK. Do not use electronic devices. (Note: if you need to do arithmetic with numbers that have more than two digits, you are on the wrong track. If you cannot solve this mod 88, solve it mod 44 (15 points).)

7. (BONUS: 8B points) Let p be an odd prime and a an integer, $a \not\equiv 0 \pmod{p}$. Prove: $(\exists x)(x^2 \equiv a \pmod{p}) \Rightarrow (\exists y)(y^2 \equiv a \pmod{p^2})$.

8. (BONUS: 6B points) Given $a_1, \ldots, a_n \in \mathbb{Z}$ decide whether or not $\bigcup_{i=1}^{n} a_i\mathbb{Z}$ is a subgroup. Give a very simple, algorithmically efficiently decidable necessary and sufficient condition. Prove your answer.