CMSC-37110 Discrete Mathematics
SECOND QUIZ December 2, 2014

Name (print): ______________________________

Do **not** *use book, notes, scratch paper.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else. **Write your solution in the space provided.** You may **continue on the reverse**. This exam contributes 5% to your course grade.

1. (4+7+7+6+4 points) A polynomial of degree at most 3 is an expression of the form $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ where $a_i \in \mathbb{R}$. Let P_3 denote the set of these polynomials. Note (do not prove) that P_3 is a vector space.

   (a) State $\dim(P_3)$ and list a basis of P_3. Do not prove.

   (b) Consider the set $T = \{f \in P_3 \mid f(1) = 0\}$. Note (do not prove) that this is a subspace of P_3. State $\dim(T)$ and list a basis of T. Do not prove.

   (c) Prove that the following four polynomials are linearly dependent: $f_1(x) = x^3 + 3x^2 + 3x + 1$, $f_2(x) = x^3 + 1$, $f_3(x) = x^2 + 3x + 2$, $f_4(x) = (x^2 - 1)(x + 17)$. Your proof should not involve any calculation.

   (d) Let U be the span of the four polynomials from part (c) and let $g(x) = x^3 + x + 1$. Prove: $g \notin U$. Your proof should not involve any calculation.

   (e) Let $D : P_3 \to P_3$ be the differentiation operator: $D(f) = f'$. (For instance, $D(f_3) = 2x+3$.) Note (do not prove) that this is a linear transformation of P_3. Find the eigenvalues and eigenvectors of D. Prove that there are no eigenvectors other than what you found.

2. (4 points) Let a_n and b_n be sequences of **positive** numbers. TRUE or FALSE (circle the right answer): $a_n = o(b_n) \implies a_n = o(b_n^2)$. (Note the little-oh notation.) If "true," do not prove. If "false," give a concrete counterexample.

3. (6 points) Define linear independence. Your definition should be the phrase *"The vectors $v_1, \ldots, v_k$ are linearly independent if"* followed by a properly quantified formula with no English words. State the formula.

4. (6 points) Compute the characteristic polynomial and the eigenvalues of the matrix

$$\begin{pmatrix} 3 & 4 \\ 2 & -1 \end{pmatrix}$$

5. (6 points) Draw the diagram (digraph with transition probabilities) of a weakly connected Markov Chain that has more than one stationary distribution. State two stationary distributions. Use as few states as possible.

6. (BONUS: 3B points) Let $A \in M_3(\mathbb{R})$ (a 3×3 matrix). Prove: A has an eigenvector in $\mathbb{R}^3$.

7. (BONUS: 4B points) Find a 3×3 stochastic matrix that has one real and two non-real complex eigenvalues.

8. (BONUS: 6B points) Let $A_1, \ldots, A_m$ be events such that $(\forall i)(P(A_i) = 1/2)$ and $(\forall i \neq j)(P(A_i \cap A_j) \leq 1/5)$. Prove: $m \leq 6$. (Hint: Let X_i be the indicator variable of A_i. Compute $\mathrm{Var}(\sum X_i)$.)