Algorithms in Finite Groups

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1 Characteristic subgroups, solvability, nilpotence

Exercise 1.1. $G' = [G, G]$ denotes the *commutator subgroup* of G. Prove:

(a) G/G' is abelian.

(b) G' is the smallest normal subgroup $N \triangleleft G$ such that G/N is abelian.

Exercise 1.2. We defined solvability via the commutator chain. Prove:
G is solvable $\iff$ all of its composition factors are abelian.

Exercise 1.3. G is a simple abelian group $\iff$ $G \cong \mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$ for some prime p.

Exercise 1.4. Conjugation by an element of G is an automorphism of G. These automorphisms are called *inner automorphisms*. They form the group $\text{Inn}(G)$. What is the kernel of the natural homormophism $G \to \text{Inn}(G)$?

Exercise 1.5. $\text{Inn}(G) \triangleleft \text{Aut}(G)$. The quotient $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$ is the *outer automorphism group.*

Exercise 1.6. Every characteristic subgroup is normal, but a normal subgroup is not necessarily characteristic. Find a very small counterexample.

Exercise 1.7. G' char G and $Z(G)$ char G.

Exercise 1.8. Being a characteristic subgroup is a transitive relation:
If H char K char G then H char G.

Exercise 1.9. Being a normal subgroup is, in general, not a transitive relation. Find a small group G with subgroups H, K such that $K \triangleleft H \triangleleft G$ but $K \ntriangleleft G$. (Hint: $|G| = 12$ suffices.)

Exercise 1.10. If H char $K \triangleleft G$ then $H \triangleleft G$.

Exercise 1.11. If $H \triangleleft G$ char K then it does not follow that $H \triangleleft G$. Find a small counterexample.

Exercise 1.12. A finite group G is characteristically simple $\iff$ there is a simple group T such that $G \cong T \times T \times \cdots \times T$.

Exercise 1.13. Let $T_1, T_2, \cdots, T_k$ be non-abelian finite simple groups. How many normal subgroups does $T_1 \times T_2 \times \cdots \times T_k$ have? (Answer: 2^k)

Exercise 1.14. How many subgroups does $\mathbb{Z}_p \times \mathbb{Z}_p$ have? What is the number of subgroups of order p^k in the elementary abelian group of order p^n (i. e., the direct product of n copies of $\mathbb{Z}_p$) ?

Exercise 1.15. For a prime power q and $n \geq k \geq 0$, the Gaussian binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ (also called "q-binomial coefficient") is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}$$

(a) Prove: $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the number of k-dimensional subspaces in the n-dimensional space over $\mathbb{F}_q$ (the field of order q).

(b) Prove: $\lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$. ("Sets are subspaces over the 1-element field.")

Exercise 1.16. The *descending central series* (or "lower central series") of the group G is defined by $K_1(G) = G$, $K_{i+1}(G) = [G, K_i(G)]$. The *ascending central series* (or "upper central series") is defined by $Z_0(G) = 1$, $Z_1(G) = Z(G)$, and $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$. We say that G is *nilpotent* if $(\exists i)(K_i(G) = 1)$. Prove: G is nilpotent $\iff$ $(\exists j)(Z_j(G) = G)$.

Exercise 1.17. G is nilpotent $\implies$ G is solvable.

Exercise 1.18. Find a very small group that is solvable but not nilpotent. (Hint: $|G| = 6$ suffices.)

Exercise 1.19. For any subset $S \subseteq G$, the number of subsets of G conjugate to S is $|G : N_G(S)|$. In particular, the number of conjugates of an element g is $|G : C_G(g)|$.

Exercise 1.20. A p-group is a group in which the order of every element is a power of p. Prove: G is a finite p-group $\iff$ $|G| = p^n$ for some n.

Exercise 1.21. (a) Every nontrivial finite p-group has nontrivial center. (Hint: partition the set $G \setminus Z(G)$ into conjugacy classes.)
(b) Every finite p-group is nilpotent.

Exercise 1.22. For any subset $S \subset G$, we have $C_G(S) \triangleleft N_G(S)$.

Exercise 1.23. If $|G| = 15$ then $|G|$ is cyclic.

Exercise 1.24. Find a nonabelian group of order 21.

Exercise 1.25. If a Sylow p-subgroup is normal in the finite group G then it is characteristic in G.

Exercise 1.26. Let G be finite. All Sylow subgroups of G are normal $\iff$ G is the direct product of its Sylow subgroups $\iff$ G is nilpotent.

Exercise 1.27. (a) If G is not abelian then $G/Z(G)$ is not cyclic.

(b) Corollary: If $|G| = p^2$ (p prime) then G is abelian.

Exercise 1.28. The mod p Heisenberg group H_p (upper triangular matrices over $\mathbb{F}_p$ with 1s in the diagonal) has exponent p for all odd primes p.

Exercise 1.29. G has exponent 2 $\implies$ G is abelian.

Exercise 1.30. Find a nonabelian group of order p^3 with exponent p^2.

Exercise 1.31. The modulo 2 Heisenberg group H_2 is isomorphic to either the quaternion group Q_8 or the dihedral group D_4 (these being the only two nonabelian groups of order 8). Which one?

Exercise 1.32. Read about the alternating group A_n.

Exercise 1.33. (a) $H \leq S_n$ with $|S_n : H| = 2 \implies H = A_n$.
(b) Corollary: A_n char S_n.

Exercise 1.34. $Z(S_n) = 1$ for $n \geq 3$.

Exercise 1.35. A_n is simple for $n \geq 5$ and solvable for $n \leq 4$.

Exercise 1.36. $\mathrm{Out}(S_n) = 1$ for $n \neq 6$. Also $|\mathrm{Out}(S_6)| = 2$.

Exercise 1.37. Let $S_n^{(2)}$ denote the image of S_n in $S_{\binom{n}{2}}$ under the induced action on pairs. Prove: for $n \geq 5$, the group $S_n^{(2)}$ is primitive.

Exercise 1.38. G is primitive and $N \triangleleft G$, $N \neq 1 \implies N$ is transitive.

Exercise 1.39. G is solvable and acts primitively on $\Omega \implies |\Omega| = p^k$ for some k and prime p.