Algorithms in Finite Groups

MATH 37500: László Babai
Scribe: Redmond McNamara

10/02/14

2 Subdirect products, graph automorphisms

Exercise 2.1. Subgroups and quotients of solvable/nilpotent groups are solvable/nilpotent, respectively.

Definition 2.2. Let $G = G_1 \times \cdots \times G_k$. Let $\pi_i : G \to G_i$ denote the i-th projection. We say that a subgroup $H \le G$ is a *subdirect product* of the G_i if $(\forall i)(\pi_i(H) = G_i)$.

Exercise 2.3. Let $H \le T_1 \times \cdots \times T_k$ be a subdirect product of the non-abelian finite simple groups T_i. Prove: H is a direct product of diagonals.

Exercise 2.4. Suppose G is transitive. Then B is a block of imprimitivity if and only if $B^g = B$ or $B^g \cap B = \emptyset$ for all $g \in G$.

Exercise 2.5. If X is a tournament then $\mathrm{Aut}(X)$ is solvable.

Exercise 2.6 (Frucht)**.** Every finite group is isomorphic to the automorphism group of a finite trivalent graph. (Trivalent: every vertex has degree 3.)

Exercise 2.7. For a graph X and an edge e of X, let $\mathrm{Aut}(X)_e$ denote the stabilizer of e in the automorphism group of X. Let X be a connected trivalent graph. Then $\mathrm{Aut}(X)_e$ is a 2-group.

Exercise 2.8. If X is a connected graph and every vertex has degree ≤ 5 then $\mathrm{Aut}(X)_e$ is solvable.

Exercise 2.9. If X is a connected graph and every vertex has degree $\le d$ then every composition factor of $\mathrm{Aut}(X)_e$ is a subgroup of S_{d-1}.

Exercise 2.10. If $G \le S_n$ is nilpotent and primitive the n is a prime number and $|G| = n$.

Exercise 2.11. G_x denotes the stabilizer of x and x^G denotes the orbit of x. Prove: $|G_x| \cdot |x^G| = |G|$.

Exercise 2.12. If G is transitive and acts on $n = p^k$ points, and if P is a Sylow p-subgroup of G then P is transitive.