Algorithms in Finite Groups

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4 Primitive groups, affine groups, order of primitive groups

Lemma 4.1. *If $G \leq S_n$ is primitive and solvable, then $n = p^k$.*

Proof. G is solvable $\implies$ $\exists N \neq 1$ such that $N \lhd G$ and $N' = 1$. $N \lhd G$ and G primitive $\implies$ N is transitive. Since N is also abelian, it implies N is regular and hence $N \lhd_{\min} G$, i.e. N is a minimal nontrivial normal subgroup of G. So N is characteristically simple, which is elementary abelian, i.e. $N \cong \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$. □

Definition 4.2. $\mathrm{AGL}(d, q)$ is the group of affine transformations $x \mapsto Ax + b$ where $A \in M_d(\mathbb{F}_q)$, $\det(A) \neq 0$ and $b \in \mathbb{F}_q^d$.

Theorem 4.3. *Let G be a primitive permutation group. If G has an (elementary) abelian normal subgroup $N \neq 1$, then $G \leq \mathrm{AGL}(k, p)$ where $p^k = |N|$.*

Theorem 4.4. *Let $G \leq \mathrm{Sym}(\Omega)$ be a permutation group, and N a regular normal subgroup. Then (1) $G = N \rtimes G_x$ and (2) $G_x \leq \mathrm{Aut}(N)$.*

Exercise 4.5. Prove conclusion (2) in the theorem.

Exercise 4.6. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $0 \leq a_n < 1$ for each n. Prove: $\prod_{n=1}^{\infty}(1 - a_n) \neq 0 \iff \sum_{n=1}^{\infty} a_n < \infty$.

Exercise 4.7. Prove: If $G \leq S_n$ has a regular normal subgroup, then $|G| \leq n^{1+\log_2 n}$.

Exercise 4.8. Let T be a tournament. Prove: $\mathrm{Aut}(T)$ has odd order.

Theorem 4.9 (Odd Order Theorem, Feit–Thompson)**.** *Every group of odd order is solvable.*

Exercise 4.10. Prove the following statements are equivalent:

1. $\mathrm{Aut}(T)$ is solvable for all tournament T.
2. Odd Order Theorem.

Definition 4.11. Γ_k is the set of finite groups that all composition factors are subgroups of S_k.

Exercise 4.12. Prove: $G \in \Gamma_k \iff G$ has a subgroup chain

$$G = G_0 \geq G_1 \geq \cdots \geq G_m = 1$$

such that $(\forall i)(|G_{i-1} : G_i| \leq k)$.

Definition 4.13. (1) $\widetilde{\Gamma}_k$ is the class of finite groups such that all nonabelian composition factors are subgroups of S_k. (2) $\overline{\Gamma}_k$ is the class of finite groups that do not have A_{k+1} as a section (quotient of subgroup).

Exercise 4.14. Prove: $\widetilde{\Gamma}_k \subset \overline{\Gamma}_k$.