Algorithms in Finite Groups

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8 Distance Regular Graphs and 2-Closed Groups

8.1 Association schemes, metric schemes

Note 8.1. Let $\mathfrak{X} = (\Omega; R_0, \ldots, R_{r-1})$ be a coherent configuration. Let A_i be the adjacency matrix of the digraph (Ω, R_i). Then

$$\sum_{i=0}^{r-1} A_i = J \text{ and } \sum_{i=0}^{r_0-1} A_i = I$$

Furthermore, $(\forall i)(\exists j)(A_i^T = A_j)$ (transpose) and for all i, j, we have

$$A_i A_j = \sum_{k=0}^{r-1} p_{ij}^k A_k. \tag{1}$$

Thus, $\mathcal{A} = \operatorname{span}(A_0, \ldots A_{r-1})$ is an algebra and the adjacency matrices form a basis for this algebra. This algebra and its representation theory are due to Donald Higman (1975). The same algebra was introduced by Boris Weisfeiler and Andrey Leman in 1968 under the name "cellular algebra."
We also note that the algebra is closed under pointwise (Hadamard) multiplication, $A_i \circ A_j = \delta_{ij} A_i$.

Definition 8.2. If, for all i, $R_i^{-1} = R_i$ then $\mathfrak{X}$ is homogeneous and $\mathfrak{X}$ is called an *association scheme.* Under these conditions, $\mathcal{A}$ is commutative.

Definition 8.3. Let X be a graph and let $R_i = \{(x, y) \mid \operatorname{dist}(x, y) = i\}$. We say that the graph X is distance regular if this is an association scheme. Such an association scheme is called a *metric scheme.*

Definition 8.4. A graph is *distance transitive* if

$$(\forall x, y, u, v)(\operatorname{dist}(x, y) = \operatorname{dist}(u, v) \implies (\exists \alpha \in \operatorname{Aut}(X))(x^\alpha = u \wedge y^\alpha = v)).$$

Exercise 8.5. A graph is distance regular of diameter 2 if and only if it is a connected, not complete strongly regular graph.

Exercise 8.6. If X is a graph of diameter d then the adjacency matrix of X has $\geq d+1$ distinct eigenvalues.

Exercise 8.7. If X is distance regular then the number of distinct eigenvalues $= d+1$.

Definition 8.8. Consider the graph on the vertex set $\binom{[v]}{k}$ where A, B are adjacent if $|A \cap B| = k-1$. This graph is distance-transitive and the corresponding metric scheme is called the *Johnston scheme*.

Definition 8.9. Consider the graph on the vertex set Σ^s of strings of length s over an alphabet Σ where two strings are adjacent if their Hamming distance is 1. This graph is distance-transitive and gives rise to a metric scheme called the *Hamming scheme*.

Definition 8.10. The *Kneser graph* has vertex set $\binom{[v]}{k}$ where A, B are adjacent if $A \cap B = \emptyset$.

Exercise 8.11. For what values of n and k is the Kneser graph distance-regular?

Definition 8.12. A configuration is *homogeneous* if $R_0 = \text{diag}(\Omega)$. A configuration is *primitive* if $(\forall i \geq 1)((\Omega, R_i)$ is strongly connected).

Exercise 8.13. If X is primitive, X is homogeneous.

Note 8.14. Recall $G \leq S_n$ is primitive if and only if $(X)(G)$ is primitive. Furthermore, recall that $G \leq \text{Aut}((X)(G))$.

8.2 2-closed pemutation groups

Definition 8.15. A group $G \leq \text{Sym}(\Omega)$ is *2-closed* if

$$(\forall \pi \in \text{Sym}(\Omega))(\text{if } ((\forall x, y)(\exists \alpha \in G)(x^\pi = x^\alpha \text{ and } y^\pi = y^\alpha)) \implies \pi \in G)$$

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Definition 8.16. The *2-closure* of G is

$$2\text{cl}(G) := \text{Aut}((X)(G)) = \bigcap \text{ of all 2-closed groups that contain } G$$

Exercise 8.17. G is 2-closed if and only if $G = \text{Aut}(X)$ for some (not necessarily coherent) configuration.

Exercise 8.18. 2-closed graphs are closed under intersection. (Proved in class: superimpose the coloring).

Exercise 8.19. Sylow p-subgroups of 2-closed groups are 2-closed.

Exercise 8.20. If G is 2-closed then both the pointwise and setwise stabilizer of any subset is 2-closed.

Exercise 8.21. Every regular permutation group is 2-closed.

Exercise 8.22. $\pi \in$ 2-closure of G iff $(\forall(x,y))(\exists\alpha \in G)(x^\pi = x^\alpha$ and $y^\pi = y^\alpha)$.

Exercise 8.23. 2-cl$(G) = S_n$ if and only if G is doubly transitive.

Exercise 8.24 (Liebeck)**.** If $\mathrm{Aut}(X) \cong A_k$ then $n \geq \frac{1}{2}\binom{k}{\lfloor k/2 \rfloor}$. In fact, if $G \leq S_n$ is 2-closed and $G \cong A_k$ then G has an orbit of length $\geq \frac{1}{2}\binom{k}{\lfloor k/2 \rfloor}$.

Definition 8.25. Let $G \leq \mathrm{Sym}(\Omega)$ and $H \leq \mathrm{Sym}(\Phi)$. We say that H is a section of G if there exists a G-invariant subset $\Phi' \subseteq \Omega$ such that the restriction of G to Φ' (as a subgroup of $\mathrm{Sym}(\Phi')$) is permutationally isomorphic to H.

Conjecture 8.26 (Babai)**.** If G is 2-closed and acts on an orbit as A_k in its natural action then G has an orbit of length $\exp(\Omega(k))$.

Exercise 8.27 (Frucht 1938)**.** For all G there exists an X such that $\mathrm{Aut}(X) \cong G$.

Definition 8.28. Let $G \leq \mathrm{Sym}(\Omega)$ and $H \leq \mathrm{Sym}(\Phi)$. We say that H is a section of G if there exists a G-invarinat subset $\Phi' \subseteq \Omega$ such that the restriction of G to Φ' (as a subgroup of $\mathrm{Sym}(\Phi')$ is permutationally isomorphic to H.

Exercise 8.29 (I. Z. Brouwer, Babai 1968)**.** If G is any permutation group then there exists a graph X such that $\mathrm{Aut}(X) \cong G$ and G is a section of $\mathrm{Aut}(X)$.

Question 8.30. Let us represent $\mathbb{Z}_2^k$ on $2k$ elements with k orbits of length 2 in the natural way. Let H_k be the subgroup of index 2 in this group corresponding to the subgroup $\{(x_1, \dots, x_k) \mid \sum x_i = 0\} \leq \mathbb{Z}^k$ (so $|H_k| = 2^{k-1}$). If $G \leq S_n$ is 2-closed and has H_k as a section then n is large, at least superpolynomial in k. The problem is open even if $H_k \cong G$.