Graph Theory – CMSC-27500
http://people.cs.uchicago.edu/∼laci/15graphs
Homework set #1. Posted 3-31.
Due Thursday, April 2, 2015 (except where otherwise stated)

Do not submit homework before its due date; it may get lost by the time we need to grade them. If you must submit early, write the early submissions on separate sheets, separately stapled; state "EARLY SUBMISSION" on the top, and send email to the instructor listing the problems you submitted early and the reason of early submission.
Read the homework instructions on the website. The instructions that follow here are only an incomplete summary.

Hand in your solutions to problems marked "HW" and "BONUS." Do not hand in problems marked "DO." Warning: the BONUS problems are underrated. If you hand in solutions to CHALLENGE problems, do so on a separate sheet. PRINT YOUR NAME ON EVERY SHEET you submit.
Request: Use LaTeX to typeset your solutions. (You may draw diagrams by hand.) This will be mandatory after the second week, starting with the solutions you hand in on April 14. Hand in your solutions on paper, do not email.

Carefully study the policy (stated on the website) on collaboration, internet use, and academic integrity. **State collaborations and sources both in your paper and in email to the instructor.**

In all problems, unless otherwise stated, we have a graph $G = (V, E)$ with n vertices and m edges. The number $n = |V|$ is called the *order* of G and and $m = |E|$ the *size* of G. We write $\deg(v)$ to denote the *degree* of vertex v. $\alpha(G)$ denotes the *independence number* of G, i. e., the size of the largest independent set. $\chi(G)$ denotes the *chromatic number* of G, i. e., the minimum number of colors required for a legal coloring of G. The graph G is *bipartite* if V can be written as $V = V_1 \cup V_2$ such that all edges go between V_1 and V_2 (there is no edge within V_1 and no edge within V_2). In other words, G is bipartite if and only if $\chi(G) \le 2$. C_k denotes the cycle of length k. The *Cartesian product* $H = G_1 \,\square\, G_2$ of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $H = (W, F)$ with vertex set $W = V_1 \times V_2$ and edges set $\{\{(u_1, u_2), (v_1, v_2)\} \mid u_1 = v_1 \text{ and } \{u_2, v_2\} \in E_2$ or $u_2 = v_2$ and $\{u_1, v_1\} \in E_1\}$. The Cartesian product of two paths is a *grid*, and the Cartesian product of two cycles is a *toroidal grid*. The $k \times \ell$ grid and the $k \times \ell$ toroidal grid each have $k \times \ell$ vertices.

1.1 DO **(Handshake Theorem)** Prove: $\sum_{v\in V} \deg(v) = 2m$.

1.2 DO: (a) Prove: if $u, v \in V$ and there is a $u - \cdots - v$ walk in G then there is a $u - \cdots - v$ path. (b) Review *equivalence relations.* (c) We say that vertex w is *accessible* from vertex v if there exists a path between v and w. Prove: accessibility is an equivalence relation. The equivalence classes of this relation are called the *connected components* of G.

1.3 DO: Prove that the two drawings of the Petersen graph, given in Examples 1.4 and 3.4 in Harju's notes, are isomorphic.

1.4 BONUS (5 points) Prove that the toroidal grid graphs $C_7 \square C_{24}$ and $C_8 \square C_{21}$ are not isomorphic.

1.5 HW (3+3+6 points) A graph is *self-complementary* if it is isomorphic to its complement. (a) Find a self-complementary graph with 4 vertices. (b) Find a self-complementary graph with 5 vertices. (c) Prove: If G is self-complementary then either n or $n-1$ is divisible by 4.
(d) DO: Prove the converse: If n or $n-1$ is divisible by 4 then there exists a self-complementary graph with n vertices.

1.6 DO: Prove: G is bipartite if and only if G contains no odd cycle.

1.7 DO: Prove: If G is bipartite then $m \leq n^2/4$.

1.8 HW, due Tuesday, April 7 (6 points) Prove: If G is triangle-free (contains no C_3) then $m \leq n^2/4$. (Hint: induction in steps of 2, reducing the case with n vertices to the case with $n-2$ vertices.)

1.9 DO: Prove: either G or its complement, $\overline{G}$, is connected.

1.10 DO: (a) Prove: $\min(\operatorname{diam}(G), \operatorname{diam}(\overline{G})) \leq 3$. In fact, if $\operatorname{diam}(G) \geq 4$ then $\operatorname{diam}(\overline{G}) \leq 2$. (b) Find G such that $\operatorname{diam}(G) = \operatorname{diam}(\overline{G}) = 3$.

1.11 DO, due Tuesday: Prove that the Petersen graph has 120 automorphisms (self-isomorphisms).

1.12 DO, due Tuesday: Prove that the Petersen graph is not Hamiltonian (does not have a Hamilton cycle). (No "elegant" proof is known but by understanding the automorphisms of the Petersen graph you can cut down on the cases.)

1.13 HW (8 points) Determine, for what pairs (k, ℓ) is the $k \times \ell$ grid Hamiltonian. Clearly state and prove your answer. (Hint: the non-Hamiltonicity proof should be one line with reference to an exercise above. You don't need to prove that exercise.)

1.14 DO, due Tuesday: Let G be a regular graph of degree $k \geq 1$ (i. e., every vertex has degree k). Prove: $\alpha(G) \leq n/2$.

1.15 DO: Prove that the independence number of the grid is $\lceil n/2 \rceil$.

1.16 DO, due Tuesday: Determine the independence number of the $k \times \ell$ toroidal grid.

1.17 DO: Review properties of asymptotic equality from the instructor's online "Discrete Math" lecture notes. Definition: Let $\{a_n\}$ and $\{b_n\}$ be sequences. We say that these sequences are *asymptotically equal* (notation: $a_n \sim b_n$) if $\lim_{n\to\infty} a_n/b_n = 1$.

1.18 DO: Let g_n denote the number of non-isomorphic graphs of order n.
(a) Prove:

$$\frac{2^{\binom{n}{2}}}{n!} \leq g_n \leq 2^{\binom{n}{2}}.$$

(b) Prove: $\log_2 g_n \sim n^2/2$.

1.19 DO: Let $d_{\max} = \max_{v\in V} \deg(v)$ denote the maximum degree.
Prove: $\chi(G) \leq 1 + d_{\max}$.

1.20 DO: Determine the chromatic number of K_n, P_n (the path of length $n-1$), C_n, the $k \times \ell$ grid ($k, \ell \geq 3$).

1.21 HW (6 points): Determine the chromatic number of the $3 \times \ell$ toroidal grid. Prove your answer.

1.22 DO, due Tuesday: For a positive integer x, let $P_G(x)$ denote the number of those functions $h : V \to \{1, 2, \ldots, x\}$ that are legal colorings of G. Recall that $P_{K_n}(x) = x(x-1)\ldots(x-n+1)$ and $P_{\overline{K_n}}(x) = x^n$. Prove: for every graph G, the function $P_G(x)$ is a polynomial. (It is called the "chromatic polynomial" of G.)

1.23 HW, due Tuesday (6 points): Prove: $\alpha(G)\chi(G) \geq n$.