Graph Theory – CMSC-27500 – Spring 2015
Instructor: Laszlo Babai
http://people.cs.uchicago.edu/∼laci/15graphs
Homework set #10. Posted 5-1, 12:30am
Due Tuesday, May 5, typeset in **LaTeX**.

Instructor will hold a **problem session** Friday, May 1, 3:30–4:30pm, in Ry-277 (optional, will help you prepare for the next quiz). Come prepared with questions.

Read the homework instructions on the website. PRINT YOUR NAME ON EVERY SHEET you submit. **Use LaTeX to typeset your solutions**. (You may draw diagrams by hand.) Hand in your solutions on paper, do not email. If you hand in solutions to CHALLENGE problems, do so on a **separate sheet**, clearly marked "CHALLENGE," and notify the instructor by email to make sure it won't be overlooked.

Carefully study the policy (stated on the website) on collaboration, internet use, and academic integrity. **State collaborations and sources both in your paper and in email to the instructor.**

Definitions, notation. As before, $G = (V, E)$ denotes a graph or digraph with n vertices and m edges. For a digraph G, we denote the **directed line-graph** of G by $\vec{L}(G)$. So the vertices of $\vec{L}(G)$ are the edges of G; and for two edges $u \to v$ and $w \to z$ of G there is an edge $(u \to v) \to (w \to z)$ in $\vec{L}(G)$ if $v = w$. For a digraph G we define the (undirected) graph $\widetilde{G}$ by ignoring the orientation of the edges of G (and removing loops and parallel edges). So for instance if T is a tournament then $\widetilde{T}$ is a complete graph. Recall that the *girth* of a graph G is the length of its shortest cycle. The *odd-girth* of G is the length of its shortest odd cycle. (So the odd-girth of a bipartite graph is ∞.)

10.1 DO: Review the proof of the Kőváry–Turán–Sós theorem: If the graph G has no 4-cycles then $m = O(n^{3/2})$. More specifically we proved that $m \le (1/2)(n^{3/2} + n)$.

10.2 HW (8 points) Prove: If the graph G does not contain $K_{2,3}$ then $m = O(n^{3/2})$. Give a specific bound like in 10.1.

10.3 DO: Let V be a set of n points in the plane. Define the *unit-distance graph* $\mathrm{UD}(V)$ as follows: V is the set of vertices; two points $u, v \in V$ are adjacent if they are at unit distance in the plane. Prove: $m = O(n^{3/2})$ (where m is the number of edges of $\mathrm{UD}(V)$).

10.4 CH (7+5+5 points) (Erdős–DeBruijn Theorem) Let k be an integer. Prove: An infinite graph G is k-colorable iff all finite subgraphs of G are k-colorable. Give three proofs: (a) From first principles, using Zorn's lemma only. (b) Using Gödel's Compactness Theorem of first-order logic. (c) Using Tychonoff's Compactness Theorem in topology.

10.5 DO (optional): Let k_r denote the chromatic number of the unit-distance graph with vertex set $\mathbb{R}^r$. (This is an infinite graph.) By the preceding exercise, k_r is the maximum chromatic number among all *finite* unit-distance graphs in $\mathbb{R}^r$. (a) Prove: $4 \le k_2 \le 7$. In other words, the unit-distance graph of the plane is 7-colorable, and there are finite subsets of the plane of which the unit-distance graph requires at least 4 colors. (In fact, there is one with only 7 points.) (Note: The exact value of k_2 is not known; the stated bounds are the best bounds known.) (b) Prove: there exists a constant C such $k_r \le C^r$. (This is easy.) (c)(Frankl–Wilson Theorem) There exists a constant $c > 1$ such that $k_r \ge c^r$. (This is hard.)

10.6 DO: Let G be a graph. Prove: if $G \not\supseteq C_4$ then $\chi(G) = O(\sqrt{n})$.

10.7 CH (8 points): Prove: if $G \not\supseteq C_5$ then $\chi(G) = O(\sqrt{n})$.

10.8 DO: Let G be a digraph. Prove: $\chi(G) \le 2^{\chi(\vec{L}(G))}$.

10.9 DO: Let G be a DAG. Prove: (a) $\vec{L}(G)$ is a DAG. (b) The undirected graph $\widetilde{\vec{L}(G)}$ is triangle free. Note: There are two kinds of tournaments on three vertices: the directed cycle and the transitive triple (ordered set). You need to show that neither of these two occur in the directed line-graph of a DAG.

10.10 DO (triangle-free graphs of large chromatic number): Prove: for every k there exists a triangle-free graph G such that $\chi(G) \ge k$. In fact, there is such a G with $n < 4^k$ vertices.

10.11 CH (8 points) (small triangle-free graphs of large chromatic number): Prove: There exists a constant C such that for every k there exists a triangle-free graph G such that $\chi(G) \ge k$ and $n \le k^C$.

10.12 DO: Let G be a DAG. Prove: the odd-girth of $\widetilde{\vec{L}(G)}$ is strictly greater than the odd-girth of $\widetilde{G}$. (Note: this is a generalization of 10.9.)

10.13 DO: Prove: for every k and g there exists a graph G of odd-girth $\geq g$ such that $\chi(G) \geq k$. Estimate the order of G (number of vertices) we get when $g = 7$. (Hint: combine problems 10.8 ad 10.12.) — Comment. For a long time it was not known whether one can also get rid of the 4-cycles. Finally Erdős proved in 1959, using the probabilistic method, that indeed one can; in fact the result stated in this exercise remains true with "girth" in place of "odd girth." The next two exercises shed some light on why this was so difficult.

10.14 DO (optional): Prove that 10.13 remains true among infinite graphs with k an infinite cardinal (while g is a positive integer). — Hint: The same proof works. In particular, 10.8, 10.9, and 10.12 remain valid for infinite graphs.

10.15 CH (Erdős–Hajnal): Prove: If the infinite graph G has uncountable chromatic number then $G \supset C_4$. In fact, $G \supset K_{m,\aleph_1}$ for every positive integer m. — Comment: This shows that Erdős's result mentioned in the comment to exercise 10.13 cannot be generalized to infinite graphs; therefore its proof requires methods that don't generalize. Note that the method with which the odd-girth result is proved in the exercises above does generalize to the infinite (see 10.14).

10.16 DO: Let G be a graph. Prove: If every vertex has degree $\geq n/2$ then G is Hamiltonian. — Hint: use the saturation method: assume G is not Hamiltonian; keep adding edges as long as you can without making the graph Hamiltonian. (You saturated the graph with respect to being non-Hamiltonian.) So now you have a non-Hamiltonian graph that will become Hamiltonian if you add any edge.

10.∞ DO: More problems to follow. Please check back later.