Graph Theory – CMSC-27500 – Spring 2015
Instructor: Laszlo Babai
http://people.cs.uchicago.edu/∼laci/15graphs
Homework set #11. Posted 5-8, 10pm
Due Tuesday, May 12, typeset in **LaTeX**.

Instructor will hold a **problem session** Monday, May 11, 4:30–5:20pm, in Ry-276 (optional, will help you prepare for the next quiz). Come prepared with questions.

Read the homework instructions on the website. PRINT YOUR NAME ON EVERY SHEET you submit. **Use LaTeX to typeset your solutions**. (You may draw diagrams by hand.) Hand in your solutions on paper, do not email. If you hand in solutions to CHALLENGE problems, do so on a **separate sheet**, clearly marked "CHALLENGE," and notify the instructor by email to make sure it won't be overlooked.

Carefully study the policy (stated on the website) on collaboration, internet use, and academic integrity. **State collaborations and sources both in your paper and in email to the instructor.**

Definitions, notation. As before, $G = (V, E)$ denotes a graph or digraph with n vertices and m edges.

We say that a graph has a property *hereditarily* if all subgraphs have the property. A property is hereditary if whenever a graph has the property, all of its subgraphs have it. For instance, k-colorability is a hereditary property, as is being C_4-free. The property that "the minimum degree is at most k" is not hereditary, but some graphs have it hereditarily. (The minimum degree of a graph is the smallest among the dgerees of its vertices.)

In the network problems below we are given a network (G, s, t, c) where $G = (V, E)$ is a digraph, $c : E \to \mathbb{R}$ is the capacity function, s is the source vertex and t is the target vertex. The network is *integral* if every capacity is an integer. A flow is *integral* if for every edge $e \in E$, the quantity x_e is an integer (every edge carries an integer amount of flow).

11.1 DO: Review the the quiz problems (posted).

11.2 DO: Assume the graph G has minimum degree $\le k$ hereditarily (i. e., every subgraph has minimum degree $\le k$). Prove: $\chi(G) \le k + 1$.

11.3 HW (6 points): Prove: If $G \not\supset K_{2,3}$ then $\chi(G) = O(\sqrt{n})$. – You may use other HW and DO exercises (present and past) without proof but with clear reference.

11.4 DO: Review the proof of the max-flow-min-cut theorem.

11.5 DO: Prove that the set of feasible flows is a (a) non-empty (b) convex (c) closed subset of $\mathbb{R}^m$.

11.6 HW (7+5 points): We are given a network as in the preamble. Assume every $s-t$ cut has infinite capacity. (a) Prove: there exists a directed $s \to \cdots \to t$ path of which every edge has infinite capacity.
(b) Prove: the values of the $s-t$ flows are unbounded (can be arbitrarily large).

11.7 DO: Prove: if an integral network has a cut of finite capacity then it has an optimum integral flow.

11.8 DO: State the four versions of Menger's theorem.

11.9 DO: Deduce the directed edge-disjoint version of Menger's Theorem from the max-flow-min-cut theorem and problem 11.7.

11.10 HW (8 points): Deduce König's Theorem from the directed vertex-disjoint version of Menger's Theorem.

11.11 DO: Deduce the directed vertex-disjoint version of Menger's Theorem from the directed edge-disjoint version.

11.∞ DO: More problems may follow, please check back later.